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**DISCRETE SYMMETRIES AND THE PROPAGATOR APPROACH
TO COUPLED FERMIONS IN QUANTUM FIELD THEORY .
GENERALITIES . THE CASE OF A SINGLE FERMION-ANTIFERMION PAIR .**

Q. Duret ^{1 2} & B. Machet ^{1 3}

Abstract: Starting from Wigner's symmetry representation theorem, we give a general account of discrete symmetries (parity P , charge conjugation C , time-reversal T), focusing on fermions in Quantum Field Theory. We provide the rules of transformation of Weyl spinors, both at the classical level (grassmannian wave functions) and quantum level (operators). Making use of Wightman's definition of invariance, we outline ambiguities linked to the notion of classical fermionic Lagrangian. We then present the general constraints cast by these transformations and their products on the propagator of the simplest among coupled fermionic system, the one made with one fermion and its antifermion. Last, we put in correspondence the propagation of C eigenstates (Majorana fermions) and the criteria cast on their propagator by C and CP invariance.

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1 Introduction

Fermions are usually treated, in most aspects of their phenomenology, as classical, though anticommuting, objects. Their Lagrangian is commonly endowed with a mass matrix though, for coupled systems ¹, this can only be a linear approximation in the vicinity of one among the physical poles of their full (matricial) propagator [1] [2]. In this perspective, the study of neutral kaons [1], and more specially of the role held, there, by discrete symmetries P , C , T and their products, has shown that subtle differences occur between the “classical” treatment obtained from a Lagrangian and a mass matrix, and the full quantum treatment dealing with their propagator. Using a classical approximation for fermions is *a priori* still more subject to caution since, in particular, their anticommutation is of quantum origin. This is why, after the work [1], we decided to perform a study of coupled fermionic systems in Quantum Field Theory, dealing especially with the propagator approach ². Treating fermions in a rigorous way is all the more important as the very nature of neutrinos, Dirac or Majorana, is still unknown, and that all theoretical results, concerning specially flavor mixing, have been mainly deduced from classical considerations.

The second and third parts of this work are dedicated to general statements concerning, first, symmetry transformations in general, then the discrete symmetries parity P , charge conjugation C , time-reversal T , and their products. It does not pretend to be original, but tries to make a coherent synthesis of results scattered in the literature. Starting from Wigner’s representation theorem [5] and Wightman’s point of view for symmetry transformations [6], we give the general rules of transformations of operators and of their hermitian conjugates by any unitary or antiunitary transformation. We then specialize to transforming Weyl spinors by P , C , T and their products, first when they are considered at the classical level (grassmanian wave functions), then at the quantum level (anticommuting operators).

The fourth part deals with the concept of invariance of a given theory. By taking the simple example of fermionic mass terms (Dirac and Majorana), we exhibit ambiguities and inconsistencies that arise in the transformations of a classical Lagrangian by antiunitary transformations. This motivates, like for neutral kaons [1], the propagator approach, which is the only safe way of deducing unambiguously the constraints cast by symmetry transformations on the Green functions of physical (propagating) particles, from which the S-matrix can be in principle reconstructed [6].

For the sake of simplicity, it is extensively investigated only in the case of the simplest among coupled fermionic systems, the one made with a single fermion and its antifermion; such a coupling, which concerns neutral particles, is indeed allowed by Lorentz invariance. This is the object of the fifth and last part of this work. We derive in full generality the constraints cast on the propagator by P , C , T , PC , PCT . We show that the physical (propagating) fermions can only be Majorana (C eigenstates) if their propagator satisfies the constraints cast by C or CP invariance.

The extension to several flavors, with its expected deeper insight into the issue of quantum mixing in connection with discrete symmetries, is currently under investigation ³.

2 Generalities

In this paper, we shall note equivalently $\xi^\alpha \xrightarrow{\mathcal{R}} (\xi^\alpha)^\mathcal{R} \equiv \mathcal{R} \cdot \xi^\alpha$, where ξ^α is a Weyl spinor (see Appendix A.1) and $\mathcal{R} \cdot \xi^\alpha$ its transformed by \mathcal{R} ; often the “ \cdot ” will be omitted such that this transformed will also be noted $\mathcal{R}\xi^\alpha$. The corresponding fermionic field operators will be put into square brackets, for example $[\xi^\alpha]$, $[\xi^\alpha]^\mathcal{R}$, the latter being the transformed of the former by the transformation \mathcal{R} . Formally $[\xi^\alpha]^\mathcal{R} = (\xi^\alpha)^\mathcal{R}$.

¹Both quarks and leptons form coupled systems through the Higgs sector.

²The propagator approach for coupled systems was initiated in [3], then applied in [4] to the case of neutral kaons. By defining the physical masses as the poles of the full (matricial) propagator, it enabled to go beyond the Wigner-Weisskopf approximation, to deal with non-hermitian Lagrangians suitable for unstable particles, and to deduce general constraints cast by discrete symmetries. This method was then refined in [1], still in the case of neutral kaons.

³Results concerning mixing at the quantum level have been obtained, by less general techniques, in [7], [8] and [9].

The transition amplitude between two fermionic states is noted $\langle \chi | \psi \rangle$; this defines a scalar product and the corresponding norm $\langle \psi | \psi \rangle$ is real positive. The scalar product satisfies

$$\langle \psi | \chi \rangle^* = \langle \chi | \psi \rangle; \quad (1)$$

we consider furthermore that representations of the Poincaré group satisfy [10]

$$\langle \psi | \chi \rangle^* = \langle \psi^* | \chi^* \rangle. \quad (2)$$

2.0.1 The symmetry representation theorem of Wigner [5]

A symmetry transformation is defined as a transformation on the states (ray representations) $\Psi \rightarrow \Psi'$ that preserve transition probabilities

$$|\langle \Psi'_1 | \Psi'_2 \rangle|^2 = |\langle \Psi_1 | \Psi_2 \rangle|^2. \quad (3)$$

The so-called “symmetry representation theorem” states ⁴:

Any symmetry transformation can be represented on the Hilbert space of physical states by an operator that is either linear and unitary, or antilinear and antiunitary.

Since we have to deal with unitary as well as antiunitary operators, it is important to state their general properties and how they operate on fermionic field operators. A unitary operator \mathcal{U} and an antiunitary operator \mathcal{A} satisfy, respectively

$$\forall \psi, \chi \quad \langle \mathcal{U}\psi | \mathcal{U}\chi \rangle = \langle \psi | \chi \rangle, \quad \langle \mathcal{A}\psi | \mathcal{A}\chi \rangle = \langle \chi | \psi \rangle = \langle \psi | \chi \rangle^*. \quad (4)$$

Both preserve the probability transition $|\langle \psi | \chi \rangle|^2 = |\langle \mathcal{U}\psi | \mathcal{U}\chi \rangle|^2 = |\langle \mathcal{A}\psi | \mathcal{A}\chi \rangle|^2$.

2.0.2 Antiunitarity and antilinearity

An antilinear operator is an operator that complex conjugates any c-number on its right

$$\mathcal{A} \text{ antilinear} \Leftrightarrow \mathcal{A}(c | \psi \rangle) = c^* \mathcal{A} | \psi \rangle. \quad (5)$$

An antiunitary operator is also antilinear. Let us indeed consider the antiunitary operator \mathcal{A} .

$\langle \mathcal{A}\psi | \mathcal{A} | \lambda \chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}\lambda \chi \rangle = \langle \lambda \chi | \psi \rangle = \lambda^* \langle \chi | \psi \rangle = \lambda^* \langle \mathcal{A}\psi | \mathcal{A} | \chi \rangle$ shows that \mathcal{A} is antilinear.

2.0.3 Unitarity and linearity

In the same way, one shows that a unitary operator is linear.

2.0.4 Symmetry transformations: Wightman’s point of view

Wightman [6] essentially deals with vacuum expectation values of strings of field operators. The transformed $\hat{\mathcal{O}}$ of an operator \mathcal{O} is defined through the transformation that changes the state ϕ into $\hat{\phi}$

$$\langle \hat{\phi} | \mathcal{O} | \hat{\phi} \rangle = \langle \phi | \hat{\mathcal{O}} | \phi \rangle \quad (6)$$

One has accordingly:

* for a unitary transformation \mathcal{U}

$$\hat{\mathcal{O}} = \mathcal{U}^{-1} \mathcal{O} \mathcal{U}, \quad (7)$$

⁴We refer the reader to [11] for a careful demonstration of this theorem.

* for a antiunitary transformation \mathcal{A} ^{5 6}

$$\begin{aligned}\hat{\mathcal{O}} &= (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger \\ &= \mathcal{A}^\dagger\mathcal{O}^\dagger(\mathcal{A}^{-1})^\dagger = \mathcal{A}^\dagger\mathcal{O}^\dagger\mathcal{A}.\end{aligned}\quad (9)$$

This is the demonstration.

* For \mathcal{U} unitary ($\mathcal{U}\mathcal{U}^\dagger = 1 = \mathcal{U}^\dagger\mathcal{U}$):

$$\langle \mathcal{U}\psi | \mathcal{O} | \mathcal{U}\chi \rangle = \langle \psi | \mathcal{U}^\dagger\mathcal{O}\mathcal{U} | \chi \rangle = \langle \psi | \mathcal{U}^{-1}\mathcal{O}\mathcal{U} | \chi \rangle, \text{ q.e.d.}$$

* For \mathcal{A} antiunitary:

- first, we demonstrate the important relation

$$\forall(\psi, \chi) \quad \langle \mathcal{A}\psi | \mathcal{A}\mathcal{O}\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \chi | \mathcal{O}^\dagger | \psi \rangle. \quad (10)$$

Indeed:

$$\langle \mathcal{A}\psi | \mathcal{A}\mathcal{O}\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}\mathcal{O} | \chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}(\mathcal{O}\chi) \rangle \stackrel{(4)}{=} \langle \mathcal{O}\chi | \psi \rangle = \langle \chi | \mathcal{O}^\dagger | \psi \rangle;$$

- one has then, in particular ⁷

$$\langle \mathcal{A}\psi | \mathcal{O} | \mathcal{A}\chi \rangle = \langle \mathcal{A}\psi | \mathcal{A}(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})\mathcal{A}^{-1} | \mathcal{A}\chi \rangle = \langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle, \quad (12)$$

which yields the desired result for $\psi = \chi$ ⁸.

According to (9), an extra hermitian conjugation occurs in the transformation of an operator by an antiunitary transformation⁹.

2.0.5 General constraints

$$\langle \hat{\phi} | \mathcal{O}^\dagger | \hat{\phi} \rangle \stackrel{(6)}{=} \langle \phi | \widehat{\mathcal{O}^\dagger} | \phi \rangle \text{ evaluates also as } \langle \hat{\phi} | \mathcal{O}^\dagger | \hat{\phi} \rangle = \langle \hat{\phi} | \mathcal{O} | \hat{\phi} \rangle^* \stackrel{(6)}{=} \langle \phi | \hat{\mathcal{O}} | \phi \rangle^* = \langle \phi | (\hat{\mathcal{O}})^\dagger | \phi \rangle, \text{ such that, comparing the two expressions one gets}$$

$$\widehat{\mathcal{O}^\dagger} = (\hat{\mathcal{O}})^\dagger, \quad (13)$$

which is a constraint that must be satisfied by any operator \mathcal{O} transformed by unitary as well as antiunitary symmetry transformations. Eq. (13) can easily be checked explicitly. $[\psi]$ being the field operator associated with the grassmanian function ψ , one has:

* for a unitary transformation \mathcal{U} :

$$\begin{aligned}\widehat{[\psi]}^\dagger &\stackrel{(7)}{=} \mathcal{U}^{-1} [\psi]^\dagger \mathcal{U}, \\ \widehat{[\psi]}^\dagger &\stackrel{(13)}{=} ([\hat{\psi}])^\dagger \stackrel{(7)}{=} (\mathcal{U}^{-1} [\psi] \mathcal{U})^\dagger \stackrel{\mathcal{U}\mathcal{U}^\dagger=1=\mathcal{U}^\dagger\mathcal{U}}{=} \mathcal{U}^{-1} [\psi]^\dagger \mathcal{U};\end{aligned}\quad (14)$$

⁵The last equality in (9) comes from the property, demonstrated by Weinberg [11], that an antiunitary operator must also satisfy the relation $\mathcal{A}\mathcal{A}^\dagger = 1 = \mathcal{A}^\dagger\mathcal{A}$ (see Appendix B). So, in particular, one has $(\mathcal{A}^{-1})^\dagger\mathcal{A}^{-1} = 1 \Rightarrow (\mathcal{A}^{-1})^\dagger = \mathcal{A}$.

⁶ Because of (9), for $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n$

$$\begin{aligned}[\mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n]^\Theta &= (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_n\mathcal{A})^\dagger = (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{A}\mathcal{A}^{-1}\mathcal{O}_2\mathcal{A}\mathcal{A}^{-1} \dots \mathcal{A}\mathcal{A}^{-1}\mathcal{O}_n\mathcal{A})^\dagger \\ &= (\mathcal{A}^{-1}\mathcal{O}_n\mathcal{A})^\dagger \dots (\mathcal{A}^{-1}\mathcal{O}_2\mathcal{A})^\dagger (\mathcal{A}^{-1}\mathcal{O}_1\mathcal{A})^\dagger \\ &= [\mathcal{O}_n]^\Theta \dots [\mathcal{O}_2]^\Theta [\mathcal{O}_1]^\Theta;\end{aligned}\quad (8)$$

antiunitarity implies that the order of operators has to be swapped when calculating the transformed of a string of operators.

⁷When the *in* and *out* states are different, one can write accordingly

$$\langle \mathcal{A}\psi | \mathcal{O} | \mathcal{A}\chi \rangle = \langle \chi | \hat{\mathcal{O}} | \psi \rangle = \langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle \quad (11)$$

The *in* and *out* states have to be swapped in the expressions on the r.h.s., ensuring that all terms in (11) are linear in ψ and antilinear in χ .

⁸One cannot use (164) to transform $\langle \chi | (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger | \psi \rangle$ into $\langle \psi | \mathcal{A}^{-1}\mathcal{O}\mathcal{A} | \chi \rangle$ because $\mathcal{A}^{-1}\mathcal{O}\mathcal{A}$ acts linearly and should thus this considered as a unitary operator.

⁹See [6], eq.(1-30).

* for a antiunitary transformation \mathcal{A} :

$$\begin{aligned} \widehat{[\psi]^\dagger} &\stackrel{(9)}{=} (\mathcal{A}^{-1} [\psi]^\dagger \mathcal{A})^\dagger = \mathcal{A}^\dagger [\psi] \mathcal{A}, \\ \widehat{[\psi]^\dagger} &\stackrel{(13)}{=} (\mathcal{A}^\dagger [\psi]^\dagger \mathcal{A})^\dagger = \mathcal{A}^\dagger [\psi] \mathcal{A}. \end{aligned} \quad (15)$$

Since $[\psi]$ and $[\psi]^\dagger$ are, respectively, associated with the grassmanian functions ψ and ψ^* , (13) also casts constraints on the transformation of grassmanian functions:

$$\widehat{\psi^*} = (\hat{\psi})^*. \quad (16)$$

3 Discrete symmetries

3.1 Parity

3.1.1 Parity transformation on grassmanian wave functions

We adopt the convention $P^2 = -1$ [12]. Then the transformation of spinors are

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{P} i\eta_{\dot{\alpha}}(-\vec{x}, t) \quad , \quad \eta_{\dot{\alpha}}(\vec{x}, t) \xrightarrow{P} i\xi^\alpha(-\vec{x}, t), \\ \xi_\alpha(\vec{x}, t) &\xrightarrow{P} -i\eta^{\dot{\alpha}}(-\vec{x}, t) \quad , \quad \eta^{\dot{\alpha}}(\vec{x}, t) \xrightarrow{P} -i\xi_\alpha(-\vec{x}, t). \end{aligned} \quad (17)$$

The parity transformed of the complex conjugates are defined [12] as the complex conjugates of the parity transformed

$$P.(\xi^\alpha)^* = (P.\xi^\alpha)^*, \quad (18)$$

this ensures in particular that the constraints (13) and (16) are satisfied. It yields

$$\begin{aligned} (\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{P} -i(\eta_{\dot{\alpha}})^*(-\vec{x}, t) \quad , \quad (\eta_{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{P} -i(\xi^\alpha)^*(-\vec{x}, t), \\ (\xi_\alpha)^*(\vec{x}, t) &\xrightarrow{P} i(\eta^{\dot{\alpha}})^*(-\vec{x}, t) \quad , \quad (\eta^{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{P} i(\xi_\alpha)^*(-\vec{x}, t). \end{aligned} \quad (19)$$

For Dirac bi-spinors (see Appendix A), one gets

$$P.\psi_D = U_P\psi_D, \quad U_P = i\gamma^0, \quad U_P^\dagger = -U_P = U_P^{-1}, \quad U_P^2 = -1, \quad U_P^\dagger U_P = 1. \quad (20)$$

3.1.2 Parity transformation on fermionic field operators

Going to field operators, one uses (7), for unitary operators

$$[\xi^\alpha]^P = P^{-1}[\xi^\alpha]P \quad (21)$$

to get

$$\begin{aligned} P^{-1}\xi^\alpha(\vec{x}, t)P &= i\eta_{\dot{\alpha}}(-\vec{x}, t) \quad , \quad P^{-1}\eta_{\dot{\alpha}}(\vec{x}, t)P = i\xi^\alpha(-\vec{x}, t), \\ P^{-1}\xi_\alpha(\vec{x}, t)P &= -i\eta^{\dot{\alpha}}(-\vec{x}, t) \quad , \quad P^{-1}\eta^{\dot{\alpha}}(\vec{x}, t)P = -i\xi_\alpha(-\vec{x}, t), \\ P^{-1}(\xi^\alpha)^\dagger(\vec{x}, t)P &= -i(\eta_{\dot{\alpha}})^\dagger(-\vec{x}, t) \quad , \quad P^{-1}(\eta_{\dot{\alpha}})^\dagger(\vec{x}, t)P = -i(\xi^\alpha)^\dagger(-\vec{x}, t), \\ P^{-1}(\xi_\alpha)^\dagger(\vec{x}, t)P &= i(\eta^{\dot{\alpha}})^\dagger(-\vec{x}, t) \quad , \quad P^{-1}(\eta^{\dot{\alpha}})^\dagger(\vec{x}, t)P = i(\xi_\alpha)^\dagger(-\vec{x}, t), \end{aligned} \quad (22)$$

which satisfies the constraint (13). The following constraint then arises

$$(P^{-1})^2 \xi^\alpha P^2 = -\xi^\alpha. \quad (23)$$

Indeed: $(P^{-1})^2 \xi^\alpha P^2 = P^{-1}(P^{-1}\xi^\alpha P)P \stackrel{(22)}{=} P^{-1}i\eta_{\dot{\alpha}}P \stackrel{linear}{=} i P^{-1}\eta_{\dot{\alpha}}P \stackrel{(22)}{=} -\xi^\alpha$.

Taking the hermitian conjugate of the first equation of the first line in (22) and comparing it with the first equation of the third line, it is also immediate to check that $(PP^\dagger)\mathcal{O}(PP^\dagger)^{-1} = \mathcal{O}$, $\mathcal{O} = \xi^\alpha \dots$, which is correct for P unitary or antiunitary.

3.2 Charge conjugation

C is the operation which transforms a particle into its antiparticle, and *vice versa*, without changing its spin and momentum (see for example [13] p.17); it satisfies $C^2 = 1$ [12]

3.2.1 Charge conjugation of grassmanian wave functions

A Dirac fermion and its charge conjugate transform alike [12] and satisfy the same equation; the charge conjugate satisfies

$$C \cdot \psi_D = V_C \overline{\psi_D}^T, \quad (24)$$

where V_C is a unitary operator

$$V_C = \gamma^2 \gamma^0, \quad (V_C)^\dagger V_C = 1 = (V_C)^2; \quad (25)$$

equivalently

$$C \cdot \psi_D = U_C \psi_D^*, \quad U_C = V_C \gamma^0 = \gamma^2, \quad U_C^\dagger U_C = 1 = -(U_C)^2. \quad (26)$$

In terms of Weyl fermions (see Appendix A), one has

$$\psi_D \equiv \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix} \xrightarrow{C} -i \begin{pmatrix} \eta^{\dot{\alpha}*} \\ \xi_\beta^* \end{pmatrix} = -i \begin{pmatrix} g^{\dot{\alpha}\dot{\beta}} \eta_{\dot{\beta}}^* \\ g_{\alpha\beta} \xi^{\beta*} \end{pmatrix} = \begin{pmatrix} -\sigma_{\dot{\alpha}\dot{\beta}}^2 \eta_{\dot{\beta}}^* \\ \sigma_{\alpha\beta}^2 \xi^{\beta*} \end{pmatrix} = \gamma^2 \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix}^* = \gamma^2 \psi_D^*, \quad (27)$$

and, so

$$\begin{aligned} \xi^\alpha &\xrightarrow{C} -i \eta^{\dot{\alpha}*}, & \eta_{\dot{\alpha}} &\xrightarrow{C} -i \xi_\alpha^*, \\ \xi_\alpha &\xrightarrow{C} -i \eta_{\dot{\alpha}}^*, & \eta^{\dot{\alpha}} &\xrightarrow{C} -i \xi^{\alpha*}. \end{aligned} \quad (28)$$

The transformation of complex conjugates fields results from the constraint (16), which imposes

$$\begin{aligned} (\xi^\alpha)^* &\xrightarrow{C} i \eta^{\dot{\alpha}}, & (\eta_{\dot{\alpha}})^* &\xrightarrow{C} i \xi_\alpha, \\ (\xi_\alpha)^* &\xrightarrow{C} i \eta_{\dot{\alpha}}, & (\eta^{\dot{\alpha}})^* &\xrightarrow{C} i \xi^{\alpha}. \end{aligned} \quad (29)$$

One can now show that (recall that $U_C^2 = -1$ from (26))

$$C \text{ unitary and linear, } C^2 = 1. \quad (30)$$

If (16) holds, the property $C^2 = 1$ can only be realized if one considers that C is a linear operator. Indeed, then, using (28) and (29), one has $C \cdot C \cdot \xi^\alpha \stackrel{(28)}{=} C \cdot (-i(\eta^{\dot{\alpha}})^*) \stackrel{\text{linear}}{=} (-i)C \cdot (\eta^{\dot{\alpha}})^* \stackrel{(29)}{=} \xi^\alpha$, which entails, as needed, $C^2 = 1$.

The only way to keep $C^2 = 1$ while having C antilinear, as (26) seems to suggest, would be to break the relation (16), in which case, the signs of (29) get swapped. Suppose indeed that we consider that C is antilinear (thus also antiunitary), and suppose that we also want to preserve the relation (16); then, (29) stays true together with (28), and, by operating a second time with C on the l.h.s. of (28) or (29), one finds that it can only satisfy $C^2 = -1$ instead of $C^2 = 1$. Among consequences, one finds that the commutation and anticommutation relations with other symmetry transformations P and T are changed¹⁰, which swaps in particular the sign of $(PCT)^2$; also, since T is antilinear and P is linear, this would make PCT linear, thus unitary. So, if we want C to be antilinear, we have to abandon (16); considering that, at the same time, the equivalent relation (13) for operators is not true either causes serious problems with Wightman's definition (6) of the transformed of an operator (see subsection 2.0.5) which has to be either unitary or antiunitary according to the Wigner's symmetry representation theorem (see subsection 2.0.1). Refusing to go along this path, we have to keep (13) while giving up (16), that is we must

¹⁰With our conventions, we have $CP = PC$, $(PC)^2 = -1$, and $(PCT)^2 = 1$.

abandon the natural correspondence $\psi \leftrightarrow [\psi]$, $\psi^* \leftrightarrow [\psi]^\dagger$ between fields and operators. This looks extremely unnatural and a price too heavy to pay; this is why we consider that the relations (16) and $C^2 = 1$ are only compatible with unitarity and linearity for C .

The question now arises whether this causes any problem or leads to contradictions, thinking in particular of (24) and (26); if one indeed considers these two equations as the basic ones defining charge conjugation, one is led to $C \cdot (\lambda \psi_D) = \lambda^* C \cdot (\psi_D)$ and that, accordingly, C acts antilinearly on wave functions. Our argumentation rests on the fact that (24) and (26) should not be considered as so. Indeed, the two conditions defining the action of C are [12]; – that a fermion and its charge conjugate should transform alike by Lorentz; – that they should satisfy the same equation. Since the Dirac equation is linear, both $\lambda C \cdot \psi_D$ and $\lambda^* C \cdot \psi_D$ satisfy the same Dirac equation as $C \cdot \psi_D$, and thus, the same equation as ψ_D . Likewise, both $\lambda C \cdot \psi_D$ and $\lambda^* C \cdot \psi_D$ transform by Lorentz as $C \cdot \psi_D$, and thus, as ψ_D . So, the two fundamental requirements concerning the charge conjugate of a Dirac fermion bring no constraint on the linearity or antilinearity of C , and this last property must be fixed by other criteria. The ones in favor of a linear action of C have been enumerated above: – to preserve the relation $C^2 = 1$; – to preserve Wightman's definition of a symmetry transformation and to stick to Wigner's symmetry representation theorem; – to preserve both relations (16) and (13); – to preserve the natural correspondence between wave functions and field operators. Our final proposition is accordingly that: despite C complex conjugates a Dirac spinor, it has to be considered as a linear and unitary operator (in particular the relation $C \cdot \lambda \psi = \lambda C \cdot \psi$ has to be imposed), and this does not depend on whether it acts on a wave function or on a field operator.

We also refer the reader to appendix D, where a careful analysis is done of the pitfalls that accompany the use of γ matrices in the expression of the discrete transformations P , C and T .

3.2.2 Charge conjugation of fermionic field operators

According to the choice of linearity and unitarity for C , the transition from (28) and (29) for grassmanian wave functions to the transformations for field operators is done according to (7) for unitary operators, through the correspondence $\mathcal{U}\psi \leftrightarrow \mathcal{U}^{-1}[\psi]\mathcal{U}$. One gets

$$\begin{aligned} C^{-1}\xi^\alpha C &= -i(\eta^{\dot{\alpha}})^\dagger, & C^{-1}\eta_{\dot{\alpha}} C &= -i(\xi_\alpha)^\dagger, \\ C^{-1}\xi_\alpha C &= -i(\eta_{\dot{\alpha}})^\dagger, & C^{-1}\eta^{\dot{\alpha}} C &= -i(\xi^\alpha)^\dagger, \\ C^{-1}(\xi^\alpha)^\dagger C &= i(\eta^{\dot{\alpha}}), & C^{-1}(\eta_{\dot{\alpha}})^\dagger C &= i(\xi_\alpha), \\ C^{-1}(\xi_\alpha)^\dagger C &= i(\eta_{\dot{\alpha}}), & C^{-1}(\eta^{\dot{\alpha}})^\dagger C &= i(\xi^\alpha). \end{aligned} \quad (31)$$

Hermitian conjugating the first equation of the first line of (31) immediately shows its compatibility with the first equation of the third line: $C^\dagger(\xi^\alpha)^\dagger(C^{-1})^\dagger = i\eta^{\dot{\alpha}} = C^{-1}(\xi^\alpha)^\dagger C \Rightarrow (\xi^\alpha)^\dagger = CC^\dagger(\xi^\alpha)^\dagger(C^{-1})^\dagger C^{-1}$, which entails $CC^\dagger = \pm 1$ which is correct for C unitary (or antiunitary). We would find an inconsistency if the sign of the last four equations was swapped.

Since C is linear, one immediately gets

$$(C^{-1})^2 \mathcal{O} C^2 = C^{-1}(C^{-1} \mathcal{O} C)C = \mathcal{O}, \mathcal{O} = \xi^\alpha \dots \quad (32)$$

3.3 PC transformation

3.3.1 PC transformation on grassmanian wave functions

Combining (17), (28) and (29), and using, when needed, the linearity of C , one gets

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{PC} \xi_\alpha^*(-\vec{x}, t), & \eta_{\dot{\alpha}}(\vec{x}, t) &\xrightarrow{PC} \eta^{\dot{\alpha}*}(-\vec{x}, t), \\ \xi_\alpha(\vec{x}, t) &\xrightarrow{PC} -\xi^{\alpha*}(-\vec{x}, t), & \eta^{\dot{\alpha}}(-\vec{x}, t) &\xrightarrow{PC} -\eta_{\dot{\alpha}}^*(-\vec{x}, t), \end{aligned} \quad (33)$$

and

$$\begin{aligned}
(\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{PC} \xi_\alpha(-\vec{x}, t) \quad , \quad (\eta_{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{PC} \eta^{\dot{\alpha}}(-\vec{x}, t), \\
(\xi_\alpha)^*(\vec{x}, t) &\xrightarrow{PC} -\xi^\alpha(-\vec{x}, t) \quad , \quad (\eta^{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{PC} -\eta_{\dot{\alpha}}(-\vec{x}, t).
\end{aligned} \tag{34}$$

One easily checks that $(PC)^2 = -1$.

Like for charge conjugation, one has

$$PC \cdot (\xi^\alpha)^* = (PC \cdot \xi^\alpha)^*. \tag{35}$$

For a Dirac fermion, one has

$$\begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix} \xrightarrow{PC} \begin{pmatrix} \xi_\alpha^* \\ \eta^{\dot{\beta}*} \end{pmatrix} = \begin{pmatrix} g_{\alpha\beta} \xi^{\beta*} \\ g^{\dot{\beta}\dot{\gamma}} \eta_{\dot{\gamma}}^* \end{pmatrix} = \begin{pmatrix} (i\sigma^2)_{\alpha\beta} \xi^{\beta*} \\ (-i\sigma^2)_{\dot{\beta}\dot{\gamma}} \eta_{\dot{\gamma}}^* \end{pmatrix} = i \begin{pmatrix} (\eta_{\dot{\alpha}})^c \\ (\xi^\beta)^c \end{pmatrix} = i\gamma^0\gamma^2 \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix}^*, \tag{36}$$

equivalently

$$PC \cdot \psi_D = V_{PC} \bar{\psi}^T = U_P V_C \bar{\psi}^T = U_{PC} \psi^* = U_P U_C \psi^*. \tag{37}$$

As we will see in subsection 3.6, Majorana fermions have PC -parity $\pm i$.

3.3.2 PC transformation on fermionic field operators

Since we have defined PC as a linear (and unitary) operator, the transitions from grassmanian wave functions to field operators goes through (7). This yields

$$\begin{aligned}
(PC)^{-1} \xi^\alpha (PC) &= \xi_\alpha^\dagger \quad , \quad (PC)^{-1} \eta_{\dot{\alpha}} (PC) = (\eta^{\dot{\alpha}})^\dagger, \\
(PC)^{-1} \xi_\alpha (PC) &= -(\xi^\alpha)^\dagger \quad , \quad (PC)^{-1} \eta^{\dot{\alpha}} (PC) = -(\eta_{\dot{\alpha}})^\dagger, \\
(PC)^{-1} (\xi^\alpha)^\dagger (PC) &= \xi_\alpha \quad , \quad (PC)^{-1} (\eta_{\dot{\alpha}})^\dagger (PC) = \eta^{\dot{\alpha}}, \\
(PC)^{-1} (\xi_\alpha)^\dagger (PC) &= -\xi^\alpha \quad , \quad (PC)^{-1} ((\eta^{\dot{\alpha}})^\dagger (PC) = -\eta_{\dot{\alpha}}.
\end{aligned} \tag{38}$$

3.4 Time-reversal

3.4.1 Time-reversal of grassmanian wave functions

The time reversed $\langle \chi(t') | \psi(t) \rangle^T$ of a transition matrix element $\langle \chi(t') | \psi(t) \rangle$, $t < t'$ is defined by $\langle \chi(t) | \psi(t') \rangle^* = \langle \psi(t') | \chi(t) \rangle$, $t > t'$; the complex conjugation is made necessary by $t < t'$ and the fact that *in* states must occur at a time smaller than *out* states; the arrow of time is not modified when one defines the time-reversed of a transition matrix element.

The operator T is accordingly antiunitary, hence antilinear:

$$\langle TA | TB \rangle = \langle B | A \rangle \Rightarrow T \text{ antiunitary}, \tag{39}$$

In Quantum Mechanics, time-reversal must change grassmanian functions into their complex conjugate (see for example the argumentation concerning Schrödinger's equation in [13]). According to [12], the *grassmanian functions* transform by time inversion according to

$$\begin{aligned}
\psi_D(\vec{x}, t) &\xrightarrow{T} T \cdot \psi_D(\vec{x}, t) = V_T \overline{\psi_D(\vec{x}, -t)}^T; \\
V_T &= i\gamma^3\gamma^1\gamma^0, \quad V_T^\dagger V_T = 1 = V_T^2, \quad V_T^\dagger = V_T = V_T^{-1},
\end{aligned} \tag{40}$$

which introduces T as antilinear when it acts on grassmanian functions. So doing, $T \cdot \psi_D$ and ψ_D satisfy time reversed equations. One also defines

$$U_T = V_T \gamma^0 = i\gamma^3\gamma^1 = -U_T^*, U_T^\dagger = U_T = U_T^{-1}, U_T^\dagger U_T = U_T^2 = 1. \tag{41}$$

$$T \cdot \psi_D = U_T \psi_D^* = i\gamma^3 \gamma^1 \psi_D^*. \quad (42)$$

This yields for Weyl fermions

$$\begin{aligned} \xi^\alpha(\vec{x}, t) &\xrightarrow{T} -i\xi_\alpha^*(\vec{x}, -t) \quad , \quad \xi_\alpha(\vec{x}, t) \xrightarrow{T} i\xi^{\alpha*}(\vec{x}, -t), \\ \eta_{\dot{\alpha}}(\vec{x}, t) &\xrightarrow{T} i\eta^{\dot{\alpha}*}(\vec{x}, -t) \quad , \quad \eta^{\dot{\alpha}}(\vec{x}, t) \xrightarrow{T} -i\eta_{\dot{\alpha}}^*(\vec{x}, -t). \end{aligned} \quad (43)$$

The constraint (16) then entails

$$\begin{aligned} (\xi^\alpha)^*(\vec{x}, t) &\xrightarrow{T} i\xi_\alpha(\vec{x}, -t) \quad , \quad (\xi_\alpha)^*(\vec{x}, t) \xrightarrow{T} -i\xi^\alpha(\vec{x}, -t), \\ (\eta_{\dot{\alpha}})^*(\vec{x}, t) &\xrightarrow{T} -i\eta^{\dot{\alpha}}(\vec{x}, -t) \quad , \quad (\eta^{\dot{\alpha}})^*(\vec{x}, t) \xrightarrow{T} i\eta_{\dot{\alpha}}(\vec{x}, -t). \end{aligned} \quad (44)$$

One has

$$T^2 = 1, \quad CT = -TC, \quad PT = TP \quad (45)$$

3.4.2 Time-reversal of fermionic field operators

The transition to field operators is done according to (9) for antiunitary transformations, through the correspondence $(\mathcal{A}\psi)^\dagger \leftrightarrow \mathcal{A}^{-1}[\psi]\mathcal{A}$, which involves an extra hermitian conjugation with respect to the transformations of grassmanian functions ([6], eq.(1-30)):

$$\begin{aligned} T^{-1}\xi^\alpha(\vec{x}, t)T &= i\xi_\alpha(\vec{x}, -t) \quad , \quad T^{-1}\eta_{\dot{\alpha}}(\vec{x}, t)T = -i\eta^{\dot{\alpha}}(\vec{x}, -t), \\ T^{-1}\xi_\alpha(\vec{x}, t)T &= -i\xi^\alpha(\vec{x}, -t) \quad , \quad T^{-1}\eta^{\dot{\alpha}}(\vec{x}, t)T = i\eta_{\dot{\alpha}}(\vec{x}, -t), \\ T^{-1}(\xi^\alpha)^\dagger(\vec{x}, t)T &= -i(\xi_\alpha)^\dagger(\vec{x}, -t) \quad , \quad T^{-1}(\xi_\alpha)^\dagger(\vec{x}, t)T = i(\xi^\alpha)^\dagger(\vec{x}, -t), \\ T^{-1}(\eta_{\dot{\alpha}})^\dagger(\vec{x}, t)T &= i(\eta^{\dot{\alpha}})^\dagger(\vec{x}, -t) \quad , \quad T^{-1}(\eta^{\dot{\alpha}})^\dagger(\vec{x}, t)T = -i(\eta_{\dot{\alpha}})^\dagger(\vec{x}, -t). \end{aligned} \quad (46)$$

Since T is antilinear, one finds immediately that, though $T^2 = 1$, one must have

$$(T^{-1})^2 \mathcal{O} T^2 = T^{-1}(T^{-1} \mathcal{O} T)T = -\mathcal{O}, \quad \mathcal{O} = \xi^\alpha \dots \quad (47)$$

3.5 PCT transformation

3.5.1 PCT operation on grassmanian wave functions

Combining the previous results, using the linearity of P and C , one gets for the grassmanian functions¹¹

$$\begin{aligned} \xi^\alpha(x) &\xrightarrow{PCT} i\xi^\alpha(-x) \quad , \quad \eta_{\dot{\alpha}}(x) \xrightarrow{PCT} -i\eta_{\dot{\alpha}}(-x), \\ \xi_\alpha(x) &\xrightarrow{PCT} i\xi_\alpha(-x) \quad , \quad \eta^{\dot{\alpha}}(x) \xrightarrow{PCT} -i\eta^{\dot{\alpha}}(-x), \\ \psi_D(x) &\xrightarrow{PCT} i\gamma^5 \psi_D(-x), \end{aligned} \quad (48)$$

where the overall sign depends on the order in which the operators act; here they are supposed to act in the order: first T , then C and last P . When acting on bi-spinors, one has $CT = -TC$ and $PT = TP$ ¹². So, using also $CP = PC$, one gets $(PCT)(PCT) = (PCT)(P(-)TC) = (PCT)(-TPC)$. $T^2 = 1$, $C^2 = 1$, $P^2 = -1$ (our choice) and $PC = CP$ entail

$$(PCT)^2 = 1. \quad (49)$$

¹¹Examples:

$PCT \cdot \xi^\alpha = PC \cdot (T \cdot \xi^\alpha) = PC \cdot (-i\xi_\alpha^*) = P \cdot (-i)C \cdot \xi_\alpha^* = (-i)P \cdot C \cdot \xi_\alpha^* = (-i)P \cdot \eta_{\dot{\alpha}} = P \cdot \eta_{\dot{\alpha}} = i\xi^\alpha$;

$PCT \cdot (\xi^\alpha)^* = PC \cdot (T \cdot (\xi^\alpha)^*) = PC \cdot (i\xi_\alpha) = P \cdot iC \cdot \xi_\alpha = iP \cdot (-i)(\eta_{\dot{\alpha}})^* = P \cdot (\eta_{\dot{\alpha}})^* = -i(\xi^\alpha)^*$.

¹²We disagree with [12] who states that T and P anticommute.

Note that, both C and T introducing complex conjugation, the latter finally disappears and PCT introduces no complex conjugation for the grassmanian functions. This is why one has

$$PCT \cdot \psi_D(x) = U_\Theta \psi_D(-x), \quad (50)$$

$$U_\Theta = U_P U_C U_T = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = i\gamma^5, \quad U_\Theta U_\Theta^\dagger = 1 = -U_\Theta^2, \quad U_\Theta^\dagger = -U_\Theta. \quad (51)$$

For the complex conjugate fields, the constraint (16) gives

$$\begin{aligned} (\xi^\alpha)^*(x) &\xrightarrow{PCT} -i(\xi^\alpha)^*(-x) \quad , \quad (\eta_{\dot{\alpha}})^*(x) \xrightarrow{PCT} i(\eta_{\dot{\alpha}})^*(-x), \\ (\xi_\alpha)^*(x) &\xrightarrow{PCT} -i(\xi_\alpha)^*(-x) \quad , \quad (\eta^{\dot{\alpha}})^*(x) \xrightarrow{PCT} i(\eta^{\dot{\alpha}})^*(-x), \\ \psi_D^* &\xrightarrow{PCT} -i\gamma^5 \psi_D^*, \end{aligned} \quad (52)$$

such that (this only occurs for P and PCT)

$$PCT \cdot (\xi^\alpha)^* = (PCT \cdot \xi^\alpha)^* \Leftrightarrow U_\Theta (\xi^\alpha)^* \equiv ((\xi^\alpha)^*)^\Theta = (U_\Theta \xi^\alpha)^* \equiv ((\xi^\alpha)^\Theta)^*. \quad (53)$$

Since P and C are unitary and T antiunitary, PCT is antiunitary, thus *antilinear*. So, despite no complex conjugation is involved $\Theta \cdot \lambda \xi^\alpha = \lambda^* \Theta \cdot \xi^\alpha$ ¹³.

3.5.2 PCT operation on fermionic field operators

Since Θ is antiunitary, one has, according to (9)

$$\begin{aligned} \Theta^{-1} \xi^\alpha(x) \Theta &= -i(\xi^\alpha)^\dagger(-x) \quad , \quad \Theta^{-1} \xi_\alpha(x) \Theta = -i(\xi_\alpha)^\dagger(-x), \\ \Theta^{-1} \eta_{\dot{\alpha}}(x) \Theta &= i(\eta_{\dot{\alpha}})^\dagger(-x) \quad , \quad \Theta^{-1} \eta^{\dot{\alpha}}(x) \Theta = i(\eta^{\dot{\alpha}})^\dagger(-x), \\ \Theta^{-1} (\xi^\alpha)^\dagger(x) \Theta &= i\xi^\alpha(-x) \quad , \quad \Theta^{-1} (\xi_\alpha)^\dagger(x) \Theta = i\xi_\alpha(-x), \\ \Theta^{-1} (\eta_{\dot{\alpha}})^\dagger(x) \Theta &= -i\eta_{\dot{\alpha}}(-x) \quad , \quad \Theta^{-1} (\eta^{\dot{\alpha}})^\dagger(x) \Theta = -i\eta^{\dot{\alpha}}(-x). \end{aligned} \quad (54)$$

and, using the antilinearity of Θ , one gets

$$(\Theta^{-1})^2 \mathcal{O} \Theta^2 = \Theta(\Theta^{-1} \mathcal{O} \Theta) \Theta = -\mathcal{O}, \mathcal{O} = \xi^\alpha \dots \quad (55)$$

3.6 Majorana fermions

A Majorana fermion is a bi-spinor which is a C eigenstate (it is a special kind of Dirac fermion with half as many degrees of freedom); since $C^2 = 1$, the only two possible eigenvalues are $C = +1$ and $C = -1$; thus, a Majorana fermions must satisfy (see (27)) one of the two possible Majorana conditions:

$$* -i\eta^{\dot{\alpha}*} = \pm \xi^\alpha \Leftrightarrow \eta^{\dot{\alpha}} = \pm(-i)\xi^{\alpha*} \Leftrightarrow \eta_{\dot{\beta}} = \pm(-i)\xi_{\beta}^*;$$

$$* -i\xi_{\beta}^* = \pm \eta_{\dot{\beta}}, \text{ which is the same condition as above;}$$

so,

$$\psi_M^\pm = \begin{pmatrix} \xi^\alpha \\ \pm(-i)\xi_\beta^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ \pm(-i)g_{\alpha\beta}\xi^{\beta*} \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ \pm\sigma_{\alpha\beta}^2 \xi^{\beta*} \end{pmatrix}; \quad (56)$$

the $+$ sign in the lower spinor corresponds to $C = +1$ and the $-$ sign to $C = -1$ ¹⁴.

¹³This is to be put in correspondence with C , which is *linear* despite complex conjugation is involved.

¹⁴*Remark:* Arguing that $(-i)(\xi_\beta)^*$ transforms like a right fermion, we can call $\omega_{\dot{\beta}} = (-i)(\xi_\beta)^*$, and the Majorana fermion ψ_M^+ rewrites $\psi_M^+ = \begin{pmatrix} \xi^\alpha \\ \omega_{\dot{\beta}} \end{pmatrix}$. If we then calculate its charge conjugate according to the standard rules (28), one gets $\psi_M^+ \xrightarrow{C} \begin{pmatrix} -i(\omega_{\dot{\alpha}})^\dagger \\ -i(\xi_\alpha)^* \end{pmatrix} \equiv \begin{pmatrix} \xi^\alpha \\ -i(\xi_\alpha)^* \end{pmatrix}$, which shows that it is indeed a $C = +1$ eigenstate. The argumentation becomes trivial if one uses for Majorana fermions the same formula for charge conjugation as the one at the extreme right of (27) for Dirac fermions $(\psi_M)^c = \gamma^2(\psi_M)^*$, $(\chi_M)^c = \gamma^2(\chi_M)^*$.

The Majorana conditions linking ξ and η are

$$\xi^\alpha \stackrel{C=\pm 1}{=} \pm(-i)(\eta^{\dot{\alpha}})^* \Leftrightarrow \eta_{\dot{\beta}} \stackrel{C=\pm 1}{=} \pm(-i)(\xi_\beta)^*; \quad (57)$$

using formulæ(28,29) for the charge conjugates of Weyl fermions, they also write

$$\xi^\alpha \stackrel{C=\pm 1}{=} \pm(\xi^\alpha)^c, \quad \eta_{\dot{\beta}} \stackrel{C=\pm 1}{=} \pm(\eta_{\dot{\beta}})^c. \quad (58)$$

A Majorana bi-spinor can accordingly also be written ¹⁵

$$\chi_M^\pm = \begin{pmatrix} \pm(-i)(\eta^{\dot{\alpha}})^* \\ \eta_{\dot{\beta}} \end{pmatrix}, \quad (60)$$

which is identical to ψ_M^\pm by the relations (57). By charge conjugation, using (28), $\psi_M^+ \xrightarrow{C} \chi_M^+, \psi_M^- \xrightarrow{C} -\chi_M^-$.

A so-called Majorana mass term writes

$$\begin{aligned} \overline{\psi}_M \psi_M &\equiv \psi_M^\dagger \gamma^0 \psi_M \equiv \pm i [-(\xi^\alpha)^*(\xi_\alpha)^* + \xi_\alpha \xi^\alpha] = \pm i [(\xi_\alpha)^*(\xi^\alpha)^* + \xi_\alpha \xi^\alpha] \\ \text{or } \overline{\psi}_M \gamma^5 \psi_M &\equiv \psi_M^\dagger \gamma^0 \gamma^5 \psi_M \equiv \mp i [(\xi^\alpha)^*(\xi_\alpha)^* + \xi_\alpha \xi^\alpha] = \mp i [(-\xi_\alpha)^*(\xi^\alpha)^* + \xi_\alpha \xi^\alpha]. \end{aligned} \quad (61)$$

Along the same lines, Majorana kinetic terms write ¹⁶ $\overline{\psi}_M \gamma^\mu \overleftrightarrow{p}_\mu \psi_M$ or $\overline{\psi}_M \gamma^\mu \gamma^5 \overleftrightarrow{p}_\mu \psi_M$; they rewrite in terms of Weyl spinors (using (162))

$$\begin{aligned} \overline{\psi}_M \gamma^\mu \overleftrightarrow{p}_\mu \psi_M &= \psi_M^\dagger \begin{pmatrix} \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} & 0 \\ 0 & \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \end{pmatrix} \psi_M \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta + (\pm(-i)(\xi_\alpha)^*)^* \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} (\pm(-i)\xi_\beta^*) \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta + \xi_\alpha \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \xi_\beta^*, \end{aligned} \quad (62)$$

and

$$\begin{aligned} \overline{\psi}_M \gamma^\mu \gamma^5 \overleftrightarrow{p}_\mu \psi_M &= \psi_M^\dagger \begin{pmatrix} \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} & 0 \\ 0 & \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \end{pmatrix} \gamma^5 \psi_M \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta - (\pm(-i)(\xi_\alpha)^*)^* \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} (\pm(-i)\xi_\beta^*) \\ &= (\xi^\alpha)^* \overleftrightarrow{(p^0 - \vec{p} \cdot \vec{\sigma})} \xi^\beta - \xi_\alpha \overleftrightarrow{(p^0 + \vec{p} \cdot \vec{\sigma})} \xi_\beta^*. \end{aligned} \quad (63)$$

A Dirac fermion can always be written as the sum of two Majorana's (the first has $C = +1$ and the second $C = -1$): $\begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} \xi^\alpha - i(\eta^{\dot{\alpha}})^* \\ -i\xi_\beta^* + \eta_{\dot{\beta}} \end{pmatrix} + \begin{pmatrix} \xi^\alpha + i(\eta^{\dot{\alpha}})^* \\ i\xi_\beta^* + \eta_{\dot{\beta}} \end{pmatrix} \right]$.

While a Dirac fermion \pm its charge conjugate is always a Majorana fermion ($C = \pm 1$), any Majorana fermion (*i.e.* a general bi-spinor which is a C eigenstate) cannot be uniquely written as the sum of a

¹⁵The Majorana spinors ψ_M^\pm and χ_M^\pm can also be written

$$\psi_M^\pm = \begin{pmatrix} \xi^\alpha \\ \pm(-i)(\xi^\alpha)^{CP} \end{pmatrix}, \quad \chi_M^\pm = \begin{pmatrix} \pm(-i)(\eta_{\dot{\beta}})^{CP} \\ \eta_{\dot{\beta}} \end{pmatrix}; \quad (59)$$

they involve one Weyl spinor and its CP conjugate (see subsection 3.3).

¹⁶One defines as usual $\psi \overleftrightarrow{\partial} \chi = \frac{1}{2}(\psi \partial \chi - (\partial \psi) \chi)$. For anticommuting fermions $[\psi, \chi]_+ = 0$, one has $\psi \overleftrightarrow{\partial} \chi = \psi \partial \chi = \chi \partial \psi = \chi \overleftrightarrow{\partial} \psi$.

given Dirac fermion \pm its charge conjugate: this decomposition is not unique. Suppose indeed that, for example, a $C = +1$ Majorana fermion is written like the sum of a Dirac fermion + its charge conjugate $\begin{pmatrix} \theta^\alpha \\ -i\theta_\beta^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha - i(\eta^\alpha)^* \\ \eta_\beta - i\xi_\beta^* \end{pmatrix}$. Since the two corresponding equations are not independent, ξ and η cannot be fixed, but only the combination $\xi^\alpha - i(\eta^\alpha)^* \sim \xi^\alpha - i\eta^\alpha$. So, infinitely many different Dirac fermions can be used for this purpose.

A Majorana fermion can always be written as the sum of a left fermion \pm its charge conjugate, or the sum of a right fermion \pm its charge conjugate. Let us demonstrate the first case only, since the second goes exactly along the same lines

$$\begin{aligned} \psi_M^\pm &= \begin{pmatrix} \xi^\alpha \\ \pm(-i)\xi_\beta^* \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \pm(-i)\xi_\beta^* \end{pmatrix} = \psi_L \pm \gamma^2 \psi_L^* = \psi_L \pm (\psi_L)^c, \\ \psi_L &= \begin{pmatrix} \xi^\alpha \\ 0 \end{pmatrix} = \frac{1+\gamma^5}{2} \psi_D. \end{aligned} \quad (64)$$

Majorana fermions have PC parity $= \pm i$. For example, $PC. \begin{pmatrix} \xi^\alpha \\ (\eta_\beta)^c \end{pmatrix} = \begin{pmatrix} \xi_\alpha^* \\ i\xi^\beta \end{pmatrix} = i\gamma^0 \begin{pmatrix} \xi^\alpha \\ (\eta_\beta)^c \end{pmatrix}$.

They are *not* PC eigenstates (an extra γ^0 comes into play in the definition of PC -parity).

4 Invariance

4.1 Wightman's point of view [6]

The invariance of a “theory” is expressed by the invariance of the vacuum and the invariance of all n -point functions; \mathcal{O} is then a product of fields at different space-time points and ($\hat{\mathcal{O}}$ being the transformed of \mathcal{O})

$$|0\rangle = |\hat{0}\rangle, \langle 0|\mathcal{O}|0\rangle = \langle 0|\hat{\mathcal{O}}|0\rangle. \quad (65)$$

* in the case of a unitary transformation \mathcal{U} ,

$$\langle 0|\mathcal{O}|0\rangle \stackrel{sym}{=} \langle 0|\mathcal{O}^U|0\rangle \stackrel{vacuum}{=} \langle 0^U|\mathcal{O}^U|0^U\rangle, \mathcal{O}^U = \mathcal{U}^{-1}\mathcal{O}\mathcal{U}; \quad (66)$$

taking the example of parity and if $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)$, one has

$\mathcal{O}^P = P^{-1}\mathcal{O}P = \phi_1(t_1, -\vec{x}_1)\phi_2(t_2, -\vec{x}_2)\dots\phi_n(t_n, -\vec{x}_n)$, such that parity invariance writes

$$\langle 0|\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)|0\rangle = \langle 0|\phi_1(t_1, -\vec{x}_1)\phi_2(t_2, -\vec{x}_2)\dots\phi_n(t_n, -\vec{x}_n)|0\rangle. \quad (67)$$

* in the case of a antiunitary transformation \mathcal{A} ,

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle &\stackrel{sym}{=} \langle 0|\mathcal{O}^A|0\rangle = \langle 0^A|\mathcal{O}^A|0^A\rangle; \\ \mathcal{O}^A &= (\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger \Rightarrow \\ \langle 0|\mathcal{O}|0\rangle &\stackrel{sym}{=} \langle 0|(\mathcal{A}^{-1}\mathcal{O}\mathcal{A})^\dagger|0\rangle = \langle 0|\mathcal{A}^{-1}\mathcal{O}\mathcal{A}|0\rangle^*; \end{aligned} \quad (68)$$

taking the example of $\Theta = PCT$, with $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)$, one has

$$\mathcal{O}^\Theta = (\Theta^{-1}\mathcal{O}\Theta)^\dagger = (\Theta^{-1}\phi_n\Theta)^\dagger \dots (\Theta^{-1}\phi_2\Theta)^\dagger (\Theta^{-1}\phi_1\Theta)^\dagger = \phi_n^\Theta \dots \phi_2^\Theta \phi_1^\Theta.$$

For fermions [6]

$$\phi(x)^\Theta \equiv \pm\phi(-x) = (\Theta^{-1}\phi(x)\Theta)^\dagger, \quad (69)$$

such that *PCT* invariance expresses as (of course the sign is unique and must be precisely determined)

$$\begin{aligned}
\langle 0 | \phi_1(x_1) \phi_2(x_2) \dots \phi_n(x_n) | 0 \rangle &\stackrel{sym}{=} \pm \langle 0 | \phi_n(-x_n) \dots \phi_2(-x_2) \phi_1(-x_1) | 0 \rangle \\
&= \pm \langle 0 | \phi_1^*(-x_1) \phi_2^*(-x_2) \dots \phi_n^*(-x_n) | 0 \rangle^* \\
&= \pm \langle 0 | (\Theta^{-1} \phi_1(x_1) \Theta) (\Theta^{-1} \phi_2(x_2) \Theta) \dots (\Theta^{-1} \phi_n(x_n) \Theta) | 0 \rangle^*.
\end{aligned} \tag{70}$$

It is enough to change $x_i \rightarrow -x_i$ and to read all Green functions from right to left instead of reading them from left to right (like Pauli).

For a general antiunitary transformation \mathcal{A} , the last line of (68) expressing the invariance also reads, since the vacuum is supposed to be invariant by \mathcal{A}^{-1} as well as by \mathcal{A} :

$$\begin{aligned}
\langle 0 | \mathcal{O} | 0 \rangle &\equiv \langle 0 | \mathcal{O} | 0 \rangle \\
&= \langle \mathcal{A}^{-1} 0 | (\mathcal{A}^{-1} \mathcal{O} \mathcal{A})^\dagger | \mathcal{A}^{-1} 0 \rangle = \langle \mathcal{A}^{-1} 0 | (\mathcal{A}^{-1} \mathcal{O} \mathcal{A}) | \mathcal{A}^{-1} 0 \rangle^* = \langle \mathcal{A}^{-1} 0 | \mathcal{A}^{-1} (\mathcal{O} 0) \rangle^*;
\end{aligned} \tag{71}$$

requesting that, for any ϕ , $\langle \phi | \mathcal{O} | \phi \rangle = \langle \phi | (\mathcal{A}^{-1} \mathcal{O} \mathcal{A})^\dagger | \phi \rangle$ would be much stronger a condition.

Wightman's expression of the invariance is weaker than requesting $\mathcal{O} = \hat{\mathcal{O}}$, since it occurs only for VEV's and not when sandwiched between any state ϕ .

4.2 The condition $\mathcal{O} = \hat{\mathcal{O}}$

It is often used to express the invariance of a theory with (Lagrangian or) Hamiltonian \mathcal{O} by the transformation under consideration:

* For unitary transformations, this condition is equivalent to

$$\mathcal{O} = \mathcal{U}^{-1} \mathcal{O} \mathcal{U} \Leftrightarrow [\mathcal{U}, \mathcal{O}] = 0; \tag{72}$$

* For antiunitary transformations it yields (we use the property that, for unitary as well as for antiunitary operators $\mathcal{U}^{-1} = \mathcal{U}^\dagger$ and $\mathcal{A}^{-1} = \mathcal{A}^\dagger$, see Footnote 5 and Appendix B)

$$\mathcal{O} = (\mathcal{A}^{-1} \mathcal{O} \mathcal{A})^\dagger = \mathcal{A}^{-1} \mathcal{O}^\dagger \mathcal{A} \Leftrightarrow \mathcal{A} \mathcal{O} = \mathcal{O}^\dagger \mathcal{A}. \tag{73}$$

Note that this is similar (apart from the exchange $\Theta \leftrightarrow \Theta^{-1}$) to the condition proposed in [14] (p.322) as the “*PCT*” theorem for any Lagrangian density $\mathcal{L}(x)$ considered as a hermitian *operator*

$$\Theta \mathcal{L}(x) \Theta^{-1} = \mathcal{L}^\dagger(-x). \tag{74}$$

So, that the Hamiltonian commutes with the symmetry transformation can eventually be accepted when this transformation is unitary (and we have already mentioned that this statement is stronger than Wightman's expression for invariance); however, when the transformation is antiunitary, one must be more careful.

Requesting that the transformed states should satisfy the same equations as the original ones is only true for unitary transformations. It is not in the case of antiunitary operations like *T* (or *PCT*) since a time reversed fermion does not satisfy the same equation as the original fermion but the time-reversed equation.

4.3 Hamiltonian - Lagrangian.

4.3.1 The case of a unitary transformation

• Invariance of the Hamiltonian:

In Quantum Mechanics, a system is said to be invariant by a unitary transformation \mathcal{U} if the transformed of the eigenstates of the Hamiltonian H have the same energies as the original states

$$H\psi = E\psi \text{ and } H\mathcal{U} \cdot \psi = E\mathcal{U} \cdot \psi; \quad (75)$$

since \mathcal{U} is unitary, it is in particular linear, such that $E\mathcal{U} \cdot \psi = \mathcal{U} \cdot E\psi = \mathcal{U} \cdot H\psi$; this is why the invariance of the theory is commonly expressed by

$$H = \mathcal{U}^{-1}H\mathcal{U} \Leftrightarrow [\mathcal{U}, H] = 0. \quad (76)$$

Defining, according to Wightman, the transformed \hat{H} of the Hamiltonian H by $\hat{H} = \mathcal{U}^{-1}H\mathcal{U}$, we see the the invariance condition (76) also rewrites $\hat{H} = H$. No special condition of reality is required for E .

• **Invariance of the Lagrangian:**

The Lagrangian approach is often more convenient in Quantum Field Theory; it determines the (classical) equations of motion, and also the perturbative expansion. The Lagrangian density $\mathcal{L}(x)$ is written $\langle \Psi(x) | L(x) | \Psi(x) \rangle$, where L is an operator and $\Psi(x)$ is a “vector” of different fields.

A reasonable definition for the invariance of the theory is that the transformed $\mathcal{U}\Psi$ of Ψ satisfies the same equation as Ψ ; since $\mathcal{L}(x)$ and $e^{i\alpha}\mathcal{L}(x)$ will provide the same (classical) dynamics, one expresses this invariance by

$$\langle \mathcal{U} \cdot \Psi(x) | L(x) | \mathcal{U} \cdot \Psi(x) \rangle = e^{i\alpha} \langle \Psi(x) | L(x) | \Psi(x) \rangle = e^{i\alpha} \mathcal{L}(x). \quad (77)$$

Due to the unitarity of \mathcal{U} , this is equivalent to $\langle \Psi(x) | \mathcal{U}^{-1}L(x)\mathcal{U} | \Psi(x) \rangle = e^{i\alpha} \langle \Psi(x) | L(x) | \Psi(x) \rangle$ or, owing to the fact that Ψ can be anything

$$L\mathcal{U} = e^{i\alpha}\mathcal{U}L. \quad (78)$$

If one applies this rule to a mass term, and consider the mass (scalar) as an operator, the unitarity of \mathcal{U} entails that a scalar as well as the associated operator should stay unchanged. This leaves only the possibility $\alpha = 0$. The condition (78) reduces accordingly to the vanishing of the commutator $[L, \mathcal{U}]$. Wightman’s definition (6) of the transformed $\hat{L} = \mathcal{U}^{-1}L\mathcal{U}$ of the operator L makes this condition equivalent to $\hat{L} = L$. No condition of reality (hermiticity) is required on L .

4.3.2 The case of antiunitary transformations

The situation is more tricky, since, in particular, the states transformed by a antiunitary transformation (for example T) do not satisfy the same classical equations as the original states (in the case of T , they satisfy the time-reversed equations).

This why it is more convenient to work with each bilinear present in the Lagrangian or Hamiltonian, which we write for example $\langle \phi | \mathcal{O} | \chi \rangle$. ϕ, χ can be fermions or bosons, \mathcal{O} a scalar, a derivative operator Taking the example of PCT , this bilinear transforms into $\langle \Theta\phi | \mathcal{O} | \Theta\chi \rangle \stackrel{(12)}{=} \langle \chi | \hat{\mathcal{O}} | \phi \rangle = \langle \chi | (\Theta^{-1}\mathcal{O}\Theta)^\dagger | \phi \rangle$.

Application: Dirac and Majorana mass terms

• Problems with a classical fermionic Lagrangian:

In view of all possible terms compatible with Lorentz invariance, we work in a basis which can accommodate, for example, both a Dirac fermion and its antiparticle. Accordingly, For a single Dirac fermion (and its antiparticle), we introduce the 4-vector of Weyl fermions

$$\psi = \begin{pmatrix} n_L \\ n_R \end{pmatrix} = \begin{pmatrix} \xi^\alpha \\ (\xi^\beta)^c \\ (\eta_{\dot{\gamma}})^c \\ \eta_{\dot{\delta}} \end{pmatrix} \equiv \begin{pmatrix} \xi^\alpha \\ -i(\eta^{\dot{\beta}})^* \\ -i(\xi_{\dot{\gamma}})^* \\ \eta_{\dot{\delta}} \end{pmatrix} \stackrel{\text{Lorentz}}{\sim} \begin{pmatrix} \xi^\alpha \\ \eta^\beta \\ \xi_{\dot{\gamma}} \\ \eta_{\dot{\delta}} \end{pmatrix}, \quad (79)$$

where $\overset{\text{Lorentz}}{\sim}$ means “transforms like (by Lorentz)”.

Let us study the transform by *PCT* of a Dirac-type mass term $m_D \xi^{\alpha*}(x) \eta_{\dot{\alpha}}(x) = \langle \xi^{\alpha}(x) | m_D | \eta_{\dot{\alpha}}(x) \rangle$ and of a Majorana-type mass term $m_M \xi^{\alpha*}(x) (\eta_{\dot{\alpha}})^c(x) = \langle \xi^{\alpha}(x) | m_M | (\eta_{\dot{\alpha}})^c(x) \rangle$.

* m_D and m_M we first consider as operators sandwiched between fermionic grassmanian functions. The two mass terms transform, respectively, into $\langle \Theta \xi^{\alpha}(x) | m_D | \Theta \eta_{\dot{\alpha}}(x) \rangle$ and $\langle \Theta \xi^{\alpha}(x) | m_M | \Theta (\eta_{\dot{\alpha}})^c(x) \rangle$. We now use (12), which transforms these two expressions into $\langle \eta_{\dot{\alpha}} | m_D^{\Theta} | \xi^{\alpha} \rangle$ and $\langle (\eta_{\dot{\alpha}})^c | m_M^{\Theta} | \xi^{\alpha} \rangle$. Since Θ is antilinear, $\Theta^{-1} m \Theta = m^* \Rightarrow m^{\Theta} \equiv (\Theta^{-1} m \Theta)^{\dagger} = m$. So the two mass terms transform, respectively, into $m_D \langle \eta_{\dot{\alpha}} | \xi^{\alpha} \rangle \equiv m_D \eta_{\dot{\alpha}}^* \xi^{\alpha}$ and $m_M \langle \eta_{\dot{\alpha}}^c | \xi^{\alpha} \rangle \equiv m_M (\eta_{\dot{\alpha}}^c)^* \xi^{\alpha}$. Notice that $\eta_{\dot{\alpha}}^* \xi^{\alpha}$ is (using anticommutation) $(-)$ the complex conjugate of $\xi^{\alpha*} \eta_{\dot{\alpha}}$ and likewise, that $(\eta_{\dot{\alpha}}^c)^* \xi^{\alpha}$ is $(-)$ the complex conjugate of $\xi^{\alpha*} \eta_{\dot{\alpha}}^c$.

The Lagrangian density also a priori involves Dirac and Majorana mass terms $\mu_D \eta_{\dot{\alpha}}^* \xi^{\alpha}$ and $\mu_M (\eta_{\dot{\alpha}}^c)^* \xi^{\alpha}$, such that *PCT* invariance requires $m_D = \mu_D$ and $m_M = \mu_M$ ¹⁷.

* If we instead consider that $m \phi^* \chi \xrightarrow{PCT} m (\Theta \phi^*) \Theta \chi$ we obtain, using (48) and (52), that the Dirac mass term transforms into $m_D (-i \xi^{\alpha*}) (-i \eta_{\dot{\alpha}})$, that is, it changes sign by *PCT*. The Majorana mass term transforms into $m_M (-i \xi^{\alpha*}) \Theta (-i \xi_{\alpha}^*) \stackrel{\text{antilin}}{=} m_M (-i \xi^{\alpha*}) (+i) \Theta \xi_{\alpha}^* = (-i \xi_{\alpha}^*) (+i) (-i \xi_{\alpha}^*) = -i \xi^{\alpha*} \xi_{\alpha}^*$, that is, unlike the Dirac mass term, the Majorana mass term does not change sign. This alternative would in particular exclude the simultaneous presence of Dirac and Majorana mass terms (necessary for the see-saw mechanism).

* Conclusion: antiunitary transformations of a classical fermionic Lagrangian are ambiguous and can lead to contradictory statements. Defining a classical fermionic Lagrangian is most probably itself problematic ¹⁸.

• Quantum (operator) Lagrangian

Dirac and Majorana mass terms write, respectively $[\xi^{\alpha}]^{\dagger} [m_D] [\eta_{\dot{\alpha}}]$ and $[\xi^{\alpha}]^{\dagger} [m_M] [\eta_{\dot{\alpha}}^c] \stackrel{(31)}{=} [\xi^{\alpha}]^{\dagger} [m_M] (-i) [\xi_{\alpha}]^{\dagger}$. Using (8), one gets $([\xi^{\alpha}]^{\dagger} [m_D] [\eta_{\dot{\alpha}}])^{\Theta} = [\eta_{\dot{\alpha}}]^{\Theta} [m_D]^{\Theta} ([\xi^{\alpha}]^{\dagger})^{\Theta} = [\eta_{\dot{\alpha}}]^{\Theta} [m_D]^{\Theta} ([\xi^{\alpha}]^{\Theta})^{\dagger} = -i [\eta_{\dot{\alpha}}] [m_D] (-i) [\xi^{\alpha}]^{\dagger}$, such that, using the anticommutation of fermionic operators, the Dirac mass term transforms by Θ into itself.

As far as the Majorana mass term is concerned, it transforms into $([\xi^{\alpha}]^{\dagger} [m_M] [\eta_{\dot{\alpha}}^c])^{\Theta} = ([\eta_{\dot{\alpha}}^c])^{\Theta} [m_M]^{\Theta} ([\xi^{\alpha}]^{\dagger})^{\Theta} = (-i [\xi_{\alpha}]^{\dagger})^{\Theta} [m_M]^{\Theta} ([\xi^{\alpha}]^{\dagger})^{\Theta}$. One uses again (8) to evaluate $(-i [\xi_{\alpha}]^{\dagger})^{\Theta} = ([\xi_{\alpha}]^{\dagger})^{\Theta} (-i)^{\Theta} = (-i) [\xi_{\alpha}]^{\dagger} (-i) = -[\xi_{\alpha}]^{\dagger}$. So, finally, the Majorana mass term transforms into $-[\xi_{\alpha}]^{\dagger} m_M (-i) [\xi^{\alpha}]^{\dagger} \stackrel{\text{anticom}}{=} -i [\xi^{\alpha}]^{\dagger} m_M [\xi_{\alpha}]^{\dagger}$, that is, like the Dirac mass term, into itself.

The same conclusions are obtained in the propagator formalism.

5 The fermionic propagator and discrete symmetries (1 fermion + its antifermion)

The fermionic propagator $\Delta(x)$ is a matrix with a Lorentz tensorial structure, the matrix elements of which are the vacuum expectation values of \mathcal{T} -products of two fermionic operators:

$$\mathcal{T} \psi(x) \chi(y) = \theta(x^0 - y^0) \psi(x) \chi(y) - \theta(y^0 - x^0) \chi(y) \psi(x); \quad (80)$$

the Lorentz indices of the two operators yield the tensorial structure of the matrix elements.

¹⁷If the Lagrangian (Hamiltonian) is furthermore real, it should match its complex conjugate (see Appendix C). The c.c. of the Dirac mass terms are $m_D^* \xi^{\alpha} \eta_{\dot{\alpha}}^* + \mu_D^* \eta_{\dot{\alpha}} \xi^{\alpha*} \stackrel{\text{anticom}}{=} -m_D^* \eta_{\dot{\alpha}}^* \xi^{\alpha} - \mu_D^* \xi^{\alpha*} \eta_{\dot{\alpha}}$ and the c.c. of the Majorana mass term are $m_M^* \xi^{\alpha} (\eta_{\dot{\alpha}}^c)^* + \mu_M^* (\eta_{\dot{\alpha}}^c) \xi^{\alpha*} \stackrel{\text{anticom}}{=} -m_M^* (\eta_{\dot{\alpha}}^c)^* \xi^{\alpha} - \mu_M^* \xi^{\alpha*} (\eta_{\dot{\alpha}}^c)$. Using (28) to replace $\eta_{\dot{\alpha}}^c$ by $(-i) \xi_{\alpha}^*$, the reality of the Lagrangian is seen to require $m_D = -\mu_D^*$ and $m_M = -\mu_M^*$.

So, combining the two, we see that a real and *PCT* invariant (classical) Lagrangian should satisfy $m_D = \mu_D$ imaginary and $m_M = \mu_M$ imaginary.

¹⁸Let us also mention the arbitrariness that results from adding to a mass matrix any vanishing anticommutator.

If, for example, one works in the fermionic basis $(\psi_1, \psi_2, \psi_3, \psi_4)$, and if $\alpha, \beta \dots$ denote their Lorentz indices, the propagator is a 4×4 matrix $\Delta(x)$ such that

$$\Delta_{ij}^{\alpha\beta}(x) = \langle \psi_i^\alpha | \Delta(x) | \psi_j^\beta \rangle = \langle 0 | \mathcal{T}(\psi_i)^\alpha \left(\frac{x}{2}\right) (\psi_j^\dagger)^\beta \left(-\frac{x}{2}\right) | 0 \rangle. \quad (81)$$

Supposing

$$\langle \psi_i^\alpha | \psi_j^\beta \rangle = \delta_{ij} \delta^{\alpha\beta}, \quad (82)$$

we shall also use the notation,

$$\begin{aligned} \Delta(x) &= \sum_{i,j} | \psi_i^\alpha \rangle \Delta_{ij}^{\alpha\beta}(x) \langle \psi_j^\beta | \\ &= \left(\begin{array}{cccc} | \psi_1^\alpha \rangle & | \psi_2^\alpha \rangle & | \psi_3^\alpha \rangle & | \psi_4^\alpha \rangle \end{array} \right) \Delta_{ij}^{\alpha\beta}(x) \begin{pmatrix} \langle \psi_1^\beta | \\ \langle \psi_2^\beta | \\ \langle \psi_3^\beta | \\ \langle \psi_4^\beta | \end{pmatrix}; \end{aligned} \quad (83)$$

since one indeed finds $\langle \psi_i^\alpha | \Delta(x) | \psi_j^\beta \rangle = \Delta_{ij}^{\alpha\beta}(x)$.

We will work hereafter in the basis (79), which includes enough degrees of freedom to describe a (Dirac) fermion + its antifermion. The corresponding fermionic propagator is then a 4×4 matrix which involves the following types of \mathcal{T} -products ¹⁹

* mass-like propagators:

$$\begin{aligned} &\langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\hat{\beta}})^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\eta_{\hat{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ (Dirac-like),} \\ &\langle 0 | \mathcal{T} (\eta_{\hat{\alpha}})^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} \eta_{\hat{\alpha}}(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ (Dirac-like),} \\ &\langle 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\hat{\beta}})^c)^\dagger(-x) | 0 \rangle, \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) (\eta_{\hat{\beta}})^\dagger(-x) | 0 \rangle \text{ (Majorana-like),} \\ &\langle 0 | \mathcal{T} (\eta_{\hat{\alpha}})^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle, \langle 0 | \mathcal{T} \eta_{\hat{\alpha}}(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ (Majorana-like);} \end{aligned}$$

* kinetic-like propagators:

$$\begin{aligned} &\langle 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ (diagonal),} \\ &\langle 0 | \mathcal{T} (\eta_{\hat{\alpha}})^c(x) ((\eta_{\hat{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} \eta_{\hat{\alpha}}(x) (\eta_{\hat{\beta}})^\dagger(-x) | 0 \rangle \text{ (diagonal),} \\ &\langle 0 | \mathcal{T} \xi^\alpha(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) (\xi^\beta)^\dagger(-x) | 0 \rangle \text{ (non-diagonal),} \\ &\langle 0 | \mathcal{T} (\eta_{\hat{\alpha}})^c(x) (\eta_{\hat{\beta}})^\dagger(-x) | 0 \rangle \text{ and } \langle 0 | \mathcal{T} \eta_{\hat{\alpha}}(x) ((\eta_{\hat{\beta}})^c)^\dagger(-x) | 0 \rangle \text{ (non-diagonal).} \end{aligned}$$

Because of electric charge conservation, some of the mixed propagators (Majorana mass terms, non-diagonal kinetic terms) will only occur for neutral fermions.

Any propagator is a non-local functional of two fields, which are evaluated at two different space-time points; a consequence is that, unlike for the Lagrangian, which is a local functional of the fields, one cannot implement constraints coming from the anticommutation of fermions. Likewise, a propagator has no hermiticity (or reality) property, and no corresponding constraint exist ²⁰. So, the only constraints that can be cast on the propagator come from discrete symmetries and their combinations: C , CP , PCT . The mass eigenstates, which are determined from the propagator are accordingly expected to be less constrained than the eigenstates of any quadratic Lagrangian ²¹.

¹⁹For the Lagrangian, the equivalent would be to consider all possible quadratic terms compatible with Lorentz invariance. Dirac as well as Majorana mass terms are allowed, and for, kinetic terms, diagonal ones, for example $\xi^{\alpha\dagger}(p^0 - \vec{p} \cdot \vec{\sigma})_{\alpha\beta} \xi^\beta$ as well as non-diagonal ones, for example $(\eta_{\hat{\alpha}})^{c\dagger}(p^0 + \vec{p} \cdot \vec{\sigma})_{\alpha\beta} \eta_{\hat{\beta}} \equiv i \xi_\alpha(p^0 + \vec{p} \cdot \vec{\sigma})_{\alpha\beta} \eta_{\hat{\beta}}$.

²⁰Only the spectral function has positivity properties.

²¹and any mass matrix, which can only be eventually introduced in a linear approximation to the inverse propagator in the vicinity of one of its poles [2].

5.1 PCT constraints

All demonstrations proceed along the following steps.

Suppose that we want to deduce *PCT* constraints for $\langle 0 | \mathcal{T} \psi(x) \chi^\dagger(-x) | 0 \rangle$. The information that we have from (54) is: there exist ϕ and ω such that $\psi(x) = \Theta \phi^\dagger(-x) \Theta^{-1}$, $\chi^\dagger(-x) = \Theta \omega(x) \Theta^{-1}$ ²², the vacuum is supposed to be invariant $| 0 \rangle = | \Theta 0 \rangle$, and Θ is antiunitary, which entails (10)²³. We have accordingly

$$\begin{aligned} \langle 0 | \mathcal{T} \psi(x) \chi^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} \Theta \phi^\dagger(-x) \Theta^{-1} \Theta \omega(x) \Theta^{-1} | 0 \rangle \\ \text{invariance of the vacuum} &\quad \langle \Theta 0 | \mathcal{T} \Theta \phi^\dagger(-x) \Theta^{-1} \Theta \omega(x) \Theta^{-1} | \Theta 0 \rangle = \langle \Theta 0 | \mathcal{T} \Theta \phi^\dagger(-x) \omega(x) \Theta^{-1} | \Theta 0 \rangle \\ &\stackrel{(10)}{=} \langle 0 | \theta(t) \omega^\dagger(x) \phi(-x) - \theta(-t) \phi(-x) \omega^\dagger(x) | 0 \rangle = - \langle 0 | \mathcal{T} \phi(-x) \omega^\dagger(x) | 0 \rangle. \end{aligned}$$

5.1.1 Constraints on mass-like terms

$$\begin{aligned} * \text{Majorana-like } \langle 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} \xi^\alpha(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\ * \text{Majorana-like } \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\xi^\beta)^\dagger(-x)) | 0 \rangle &= \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) ((\xi^\beta)^\dagger(x)) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\xi^\beta)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x)) | 0 \rangle; \\ * \text{Majorana-like } \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\ * \text{Majorana-like } \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 \rangle; \\ * \text{Dirac-like } \langle 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} \xi^\alpha(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) \xi^\alpha(-x) | 0 \rangle; \\ * \text{Dirac-like } \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\xi^\beta)^\dagger(-x)) | 0 \rangle &= \langle 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) ((\xi^\beta)^\dagger(x)) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\xi^\beta)^\dagger(x) \eta_{\dot{\alpha}}(-x)) | 0 \rangle; \\ * \text{Dirac-like } \langle 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) (\xi^\alpha)^c(-x) | 0 \rangle; \\ * \text{Dirac-like } \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) ((\xi^\beta)^c)^\dagger(x) | 0 \rangle \\ &= - \langle 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 \rangle. \end{aligned} \tag{84}$$

We give the demonstration of the first (Majorana-like) line of (84).

$$\begin{aligned} \langle 0 | \mathcal{T} \xi^\alpha(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 \rangle &= \langle 0 | \mathcal{T} \xi^\alpha(x) i \xi_\beta(-x) | 0 \rangle = i \langle 0 | \mathcal{T} \xi^\alpha(x) \xi_\beta(-x) | 0 \rangle \\ &= i \langle 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) \Theta^{-1} \Theta(-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | 0 \rangle \\ \text{invariance of the vacuum} &\quad i \langle \Theta 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) \Theta^{-1} \Theta(-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | \Theta 0 \rangle \\ &= i \langle \Theta 0 | \mathcal{T} \Theta(-i(\xi^\alpha)^\dagger(-x)) (-i(\xi_\beta)^\dagger(x)) \Theta^{-1} | \Theta 0 \rangle \\ &= -i \langle \Theta 0 | \mathcal{T} \Theta(\xi^\alpha)^\dagger(-x) (\xi_\beta)^\dagger(x) \Theta^{-1} | \Theta 0 \rangle \\ \text{antiunitarity}^{(10)} &\quad -i \langle 0 | \theta(t) \xi_\beta(x) (\xi^\alpha)(-x) | 0 \rangle + i \langle 0 | \theta(-t) \xi^\alpha(-x) \xi_\beta(x) | 0 \rangle \\ &= +i \langle 0 | \mathcal{T} \xi^\alpha(-x) \xi_\beta(x) | 0 \rangle = \langle 0 | \mathcal{T} \xi^\alpha(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 \rangle. \end{aligned}$$

All these propagators are accordingly left invariant²⁴ by the 4-inversion $x \rightarrow -x$, or, in Fourier space, they are invariant when $p_\mu \rightarrow -p_\mu$.

²²For example, from (54), one gets $\xi^\alpha = \Theta(-i(\xi^\alpha)^\dagger) \Theta^{-1}$.

²³ Θ , though antiunitary, does not act on the θ functions of the \mathcal{T} -product because they are real.

²⁴This is not much information, but it is correct. Consider indeed the usual Feynman propagator in Fourier space for a Dirac fermion with mass m

$$\int d^4 x e^{i p x} \langle 0 | \mathcal{T} \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{pmatrix} (x) \begin{pmatrix} (\xi^\beta)^\dagger & (\eta_{\dot{\beta}})^\dagger \end{pmatrix} (-x) \gamma^0 | 0 \rangle = \frac{p_\mu \gamma^\mu + m}{p^2 - m^2} = \frac{1}{p^2 - m^2} \begin{pmatrix} m & p_\mu \overline{\sigma}^\mu \\ p_\mu \sigma^\mu & m \end{pmatrix}; \tag{85}$$

5.1.2 Constraints on kinetic-like terms

$$\begin{aligned}
* \text{ Diagonal } < 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} \xi^\alpha(-x) (\xi^\beta)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} (\xi^\beta)^\dagger(x) \xi^\alpha(-x) | 0 >; \\
* \text{ Diagonal } < 0 | \mathcal{T} (\xi^\alpha)^c(x) ((\xi^\beta)^c)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\xi^\beta)^c)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} (\xi^\beta)^c(x) (\xi^\alpha)^c(-x) | 0 >; \\
* \text{ Diagonal } < 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 >; \\
* \text{ Diagonal } < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 >; \\
* \text{ Non-diagonal } < 0 | \mathcal{T} \xi^\alpha(x) ((\xi^\beta)^c)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} \xi^\alpha(-x) ((\xi^\beta)^c)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} ((\xi^\beta)^c)^\dagger(x) \xi^\alpha(-x) | 0 >; \\
* \text{ Non-diagonal } < 0 | \mathcal{T} (\xi^\alpha)^c(x) (\xi^\beta)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} (\xi^\alpha)^c(-x) (\xi^\beta)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} (\xi^\beta)^\dagger(x) (\xi^\alpha)^c(-x) | 0 >; \\
* \text{ Non-diagonal } < 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} (\eta_{\dot{\alpha}})^c(-x) (\eta_{\dot{\beta}})^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} (\eta_{\dot{\beta}})^\dagger(x) (\eta_{\dot{\alpha}})^c(-x) | 0 >; \\
* \text{ Non-diagonal } < 0 | \mathcal{T} \eta_{\dot{\alpha}}(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 > &= - < 0 | \mathcal{T} \eta_{\dot{\alpha}}(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 > \\
&= < 0 | \mathcal{T} ((\eta_{\dot{\beta}})^c)^\dagger(x) \eta_{\dot{\alpha}}(-x) | 0 >.
\end{aligned} \tag{87}$$

In Fourier space, all these propagators must accordingly be odd in p_μ . We check like above on the Dirac propagator that it is indeed the case. One gets for example (the γ^0 in (85) now makes $\gamma_{\alpha,\beta+2}^\mu$ appear)

$$\int d^4x e^{ipx} < 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 > = \frac{p_\mu \gamma_{\alpha\beta+2}^\mu + m \delta_{\alpha\beta+2}}{p^2 - m^2}, \alpha, \beta = 1, 2, \tag{88}$$

in which only the terms linear in p_μ are present, which are indeed odd in p_μ as predicted by *PCT* invariance.

Note that *PCT* invariance does not forbid non-diagonal kinetic-like propagators.

5.1.3 Simple assumptions and consequences

PCT symmetry constrains, in Fourier space, all mass-like propagators to be p -even and all kinetic-like propagators to be p -odd; the former can only write $f(p^2) \delta_{\alpha\beta}$ and the latter $g(p^2) p_\mu \sigma_{\alpha\beta}^\mu$ or $h(p^2) p_\mu \bar{\sigma}_{\alpha\beta}^\mu$. This is what we will suppose hereafter, and consider, in Fourier space, a propagator

$$\Delta(p) = \left(\begin{array}{c} | \xi^\alpha > \quad | (\xi^\alpha)^c > \quad | (\eta_{\dot{\alpha}})^c > \quad | \eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha_1(p^2) & a_1(p^2) \\ b_1(p^2) & \beta_1(p^2) \end{array} \right) p_\mu \bar{\sigma}_{\alpha\beta}^\mu & \left(\begin{array}{cc} m_{L1}(p^2) & \mu_1(p^2) \\ m_1(p^2) & m_{R1}(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_{L2}(p^2) & m_2(p^2) \\ \mu_2(p^2) & m_{R2}(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \beta_2(p^2) & b_2(p^2) \\ a_2(p^2) & \alpha_2(p^2) \end{array} \right) p_\mu \sigma_{\alpha\beta}^\mu \end{array} \right) \left(\begin{array}{c} < \xi^\beta | \\ < (\xi^\beta)^c | \\ < (\eta_{\dot{\beta}})^c | \\ < \eta_{\dot{\beta}} | \end{array} \right). \tag{89}$$

it yields in particular (the γ^0 in (85) makes $\gamma_{\alpha,\beta}^\mu$ appear)

$$\int d^4x e^{ipx} < 0 | \mathcal{T} \xi^\alpha(x) \eta_{\dot{\beta}}(-x) | 0 > = \frac{p_\mu \gamma_{\alpha\beta}^\mu + m \delta_{\alpha\beta}}{p^2 - m^2}, \alpha, \beta = 1, 2. \tag{86}$$

PCT invariance tells us that, in a Dirac mass-like propagator, the p^μ term is not present, and the remaining term is diagonal in α, β ; and, indeed, $\gamma_{\alpha\beta}^\mu$ vanishes $\forall \alpha, \beta = 1, 2$, while the term proportional to m is diagonal in α, β .

This ansatz enables to get explicit constraints on the propagator. It is motivated by the fact that, classically, the (quadratic) Lagrangian, which is the inverse propagator, has this same Lorentz structure

$$L = \left(\begin{array}{c|c} K_1(p_-)_{\alpha\beta} & M_1 \delta_{\alpha\beta} \\ \hline M_2 \delta_{\alpha\beta} & K_2(p_+)_{\alpha\beta} \end{array} \right). \quad (90)$$

An important property is that it automatically satisfies the *PCT* constraints (84) (87). For mass-like propagators, which are invariant by the 4-inversion $x \rightarrow -x$ it is a triviality; for kinetic like propagators, the “ $-$ ” signs which occur in the r.h.s.’s of (87) are canceled by the one which comes from the differential operator p_μ acting on $(-x)$ instead of x . We consider accordingly that (89) expresses the invariance of the propagator by *PCT*.

From now onwards we shall always use the form (89) for the propagator, considering therefore that it is *PCT* invariant. It includes sixteen complex parameters. We will see how individual discrete symmetries and their products reduce this number.

5.2 Charge conjugate fields

By using the definitions of charge conjugate fields

$$\begin{aligned} \xi^\alpha &= g^{\alpha\gamma} \xi_\gamma = -i\sigma_{\alpha\gamma}^2 \xi_\gamma = -i\sigma_{\alpha\gamma}^2 (-i)((\eta_\gamma)^c)^\dagger = -\sigma_{\alpha\gamma}^2 ((\eta_\gamma)^c)^\dagger, \\ \eta_{\dot{\beta}} &= g_{\beta\delta} \eta^{\dot{\delta}} = i\sigma_{\beta\delta}^2 \eta^{\dot{\delta}} = i\sigma_{\beta\delta}^2 (-i)((\xi^\delta)^c)^\dagger = \sigma_{\beta\delta}^2 ((\xi^\delta)^c)^\dagger. \end{aligned} \quad (91)$$

one can bring additional constraints to the ones obtained from expressing the invariance by a discrete symmetry like *PCT*. We first give the example of a Dirac-like propagator:

$$\begin{aligned} < 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 > = < 0 | \mathcal{T} (-) \sigma_{\alpha\gamma}^2 ((\eta_\gamma)^c)^\dagger(x) \left(\sigma_{\beta\delta}^2 ((\xi^\delta)^c)^\dagger(-x) \right)^\dagger | 0 > \\ &= \sigma_{\alpha\gamma}^2 \sigma_{\beta\delta}^2 < 0 | \mathcal{T} ((\eta_\gamma)^c)^\dagger(x) (\xi^\delta)^c(-x) | 0 > = (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\delta\gamma}) < 0 | \mathcal{T} ((\eta_\gamma)^c)^\dagger(x) (\xi^\delta)^c(-x) | 0 > = \\ &= - < 0 | \mathcal{T} (\xi^\alpha)^c(-x) ((\eta_{\dot{\beta}})^c)^\dagger(x) | 0 > + \delta_{\alpha\beta} < 0 | \mathcal{T} (\xi^\gamma)^c(-x) ((\eta_\gamma)^c)^\dagger(x) | 0 >. \end{aligned}$$

The r.h.s. of the corresponding *PCT* constraint in the first line of (84) writes the same but for the exchange $x \rightarrow (-x)$. If we now use the ansatz (89) which implements *PCT* invariance, one gets

$$\mu_1(p^2) \delta_{\alpha\beta} = -(\delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}) m_1(p^2) \delta_{\delta\gamma} = \delta_{\alpha\beta} m_1(p^2), \quad (92)$$

equivalently

$$m_1(p^2) = \mu_1(p^2). \quad (93)$$

Likewise, one gets $m_2(p^2) = \mu_2(p^2)$.

For Majorana-like propagator, using the definitions (91) of charge conjugate fields, one gets

$$\begin{aligned} < 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}}^c)^\dagger(-x) | 0 > &= < 0 | \mathcal{T} (\eta_{\dot{\beta}}^c)^\dagger(x) \xi^\alpha(-x) | 0 > - \delta_{\alpha\beta} < 0 | \mathcal{T} (\eta_{\dot{\gamma}}^c)^\dagger(x) \xi^\gamma(-x) | 0 > \\ &= - < 0 | \mathcal{T} \xi^\alpha(-x) (\eta_{\dot{\beta}}^c)^\dagger(x) | 0 > + \delta_{\alpha\beta} < 0 | \mathcal{T} \xi^\gamma(-x) (\eta_{\dot{\gamma}}^c)^\dagger(x) | 0 >, \end{aligned} \quad (94)$$

while, with the same procedure, its transformed by *PCT* in the r.h.s. of (84) becomes

$$- < 0 | \mathcal{T} (\eta_{\dot{\beta}}^c)^\dagger(x) \xi^\alpha(-x) | 0 > = - < 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}}^c)^\dagger(-x) + \delta_{\alpha\beta} < 0 | \mathcal{T} \xi^\gamma(x) (\eta_{\dot{\gamma}}^c)^\dagger(-x) | 0 >. \quad (95)$$

One only gets tautologies such that no additional constraint arises.

We implement the same procedure for kinetic-like terms, for example $< 0 | \mathcal{T} \xi^\alpha(x) (\xi^\beta)^\dagger(-x) | 0 > = < 0 | \mathcal{T} (\xi^\beta)^\dagger(x) \xi^\alpha(-x) | 0 >$. Using $\xi^\alpha = -\sigma_{\alpha\gamma}^2 ((\eta_\gamma)^c)^\dagger$ and $(\xi^\beta)^\dagger = \sigma_{\beta\delta}^2 (\eta_{\dot{\delta}})^c$ and (89), one gets

$$\alpha_1(p^2) p_\mu \overline{\sigma^\mu}_{\alpha\beta} = -(\delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}) \beta_2(p^2) p_\mu \sigma^\mu_{\delta\gamma}$$

$$\begin{aligned}
&= -\beta_2(p^2)(p_\mu \sigma^\mu_{\alpha\beta} - \delta_{\alpha\beta} p_\mu \text{Tr} \sigma_\mu) \\
&= -\beta_2(p^2)(p_\mu \sigma^\mu_{\alpha\beta} - \delta_{\alpha\beta}(2p_0 + 0 \times p^i)) \\
&= -\beta_2(p^2)(-p_0 \sigma^0_{\alpha\beta} + \vec{p} \cdot \vec{\sigma}_{\alpha\beta}) \\
&= \beta_2(p^2) p_\mu \bar{\sigma}^\mu_{\alpha\beta},
\end{aligned} \tag{96}$$

which entails

$$\alpha_1(p^2) = \beta_2(p^2). \tag{97}$$

Likewise, one gets $\alpha_2(p^2) = \beta_1(p^2)$, and, for the non-diagonal kinetic-like propagators, $a_1(p^2) = a_2(p^2)$, $b_1(p^2) = b_2(p^2)$.

So, after making use of the definition of charge conjugate fields, (89) expressing the PCT invariance of the propagator rewrites

$$\Delta_{PCT}(p) = \left(\begin{array}{c} | \xi^\alpha > \quad | (\xi^\alpha)^c > \quad | (\eta_{\dot{\alpha}})^c > \quad | \eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \bar{\sigma}^\mu_{\alpha\beta} & \left(\begin{array}{cc} m_{L1}(p^2) & \mu_1(p^2) \\ \mu_1(p^2) & m_{R1}(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_{L2}(p^2) & \mu_2(p^2) \\ \mu_2(p^2) & m_{R2}(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & v(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} < \xi^\beta | \\ < (\xi^\beta)^c | \\ < (\eta_{\dot{\beta}})^c | \\ < \eta_{\dot{\beta}} | \end{array} \right). \tag{98}$$

PCT symmetry has finally reduced the total number of arbitrary functions necessary to describe one flavor of fermions from 16 to 10.

5.3 C constraints

C is a unitary operator and we may use directly (31) in the expression of the propagator. This is an example of demonstration, in which we suppose that the vacuum is invariant by C .

$$\begin{aligned}
&< 0 | \mathcal{T} \xi^\alpha(x) (\eta_{\dot{\beta}})^\dagger(-x) | 0 > = < C 0 | \mathcal{T} C(-i(\eta^{\dot{\alpha}})^\dagger)(x) C^{-1} C(i\xi_\beta)(-x) C^{-1} | C 0 > \\
&= < C 0 | \mathcal{T} C(\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) C^{-1} | C 0 > = < 0 | \mathcal{T} C^\dagger C(\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) C^{-1} C | 0 > \\
&= < 0 | \mathcal{T} (\eta^{\dot{\alpha}})^\dagger(x) \xi_\beta(-x) | 0 > = < 0 | \mathcal{T} ((\xi^\alpha)^c)^\dagger(x) ((\eta_{\dot{\beta}})^c)^\dagger(-x) | 0 >.
\end{aligned}$$

By using (89) expressing PCT invariance, one gets accordingly

$$\Delta_{C+PCT}(p) = \left(\begin{array}{c} | \xi^\alpha > \quad | (\xi^\alpha)^c > \quad | (\eta_{\dot{\alpha}})^c > \quad | \eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \bar{\sigma}^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \beta(p^2) & b(p^2) \\ b(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} < \xi^\beta | \\ < (\xi^\beta)^c | \\ < (\eta_{\dot{\beta}})^c | \\ < \eta_{\dot{\beta}} | \end{array} \right). \tag{99}$$

All 2×2 submatrices are in particular symmetric.

Combining now (98) and (99), a $C + PCT$ invariant propagator, after using the definition of charge conjugate fields, can finally be reduced to

$$\Delta_{C+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma}^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right), \quad (100)$$

in which the number of arbitrary functions has now been reduced to 6.

5.4 P constraints

In momentum space, the parity transformed of $p_\mu \sigma^\mu \equiv (p_0 \sigma^0 + \vec{p} \cdot \vec{\sigma})$ is $(p_0 \sigma^0 - \vec{p} \cdot \vec{\sigma}) \equiv p_\mu \overline{\sigma}^\mu$.

Using (22) and the assumption (89) expressing PCT invariance, and supposing the vacuum invariant by parity, one gets

$$\Delta_{P+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ b(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma}^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \beta(p^2) & b(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right). \quad (101)$$

A $P + C + PCT$ invariant propagator writes

$$\Delta_{P+C+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma}^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right). \quad (102)$$

The expressions above can be further reduced by using the definition of charge conjugate fields, which leads to (98) as the expression of PCT invariance. So doing, a $P + PCT$ invariant propagator writes

$$\Delta_{P+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ b(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & b(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma}^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right); \quad (103)$$

and one finds again the expression (102) for a $P + C + PCT$ invariant propagator.

5.5 CP constraints

Using (33), (89), and supposing the vacuum invariant by CP , one gets

$$\Delta_{CP+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} & \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ m(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ m(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ v(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right). \quad (104)$$

It can be further constrained by using the definition of charge conjugate fields which makes the PCT constraint be (98), to

$$\Delta_{CP+PCT}(p) = \left(\begin{array}{c} |\xi^\alpha > \quad |(\xi^\alpha)^c > \quad |(\eta_{\dot{\alpha}})^c > \quad |\eta_{\dot{\alpha}} > \end{array} \right) \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} & \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ \mu(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_L(p^2) & \mu(p^2) \\ \mu(p^2) & m_R(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & u(p^2) \\ u(p^2) & \beta(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} \end{array} \right) \left(\begin{array}{c} <\xi^\beta | \\ <(\xi^\beta)^c | \\ <(\eta_{\dot{\beta}})^c | \\ <\eta_{\dot{\beta}} | \end{array} \right). \quad (105)$$

One then gets 4 symmetric 2×2 sub-blocks.

5.6 Eigenstates of a $C + PCT$ invariant propagator

We do not consider any PCT violation, because, if this occurred, the very foundations of local Quantum Field Theory would be undermined, and the meaning of our conclusions itself could thus strongly be cast in doubt.

We look here for the eigenstates of the 4×4 matrix in (100)

$$\Delta_{C+PCT}(p^2) = \left(\begin{array}{c|c} \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} & \left(\begin{array}{cc} \rho(p^2) & \mu(p^2) \\ \mu(p^2) & \rho(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} \sigma(p^2) & m(p^2) \\ m(p^2) & \sigma(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} \alpha(p^2) & a(p^2) \\ a(p^2) & \alpha(p^2) \end{array} \right) p_\mu \overline{\sigma^\mu}_{\alpha\beta} \end{array} \right). \quad (106)$$

The three symmetric matrices $\begin{pmatrix} \rho & \mu \\ \mu & \rho \end{pmatrix}$, $\begin{pmatrix} \sigma & m \\ m & \sigma \end{pmatrix}$ and $\begin{pmatrix} \alpha & a \\ a & \alpha \end{pmatrix}$ can be simultaneously diagonalized by a unitary matrix U according to

$$\begin{aligned}
U^T \begin{pmatrix} \rho & \mu \\ \mu & \rho \end{pmatrix} U &= \begin{pmatrix} (\rho + \mu)e^{2i\varphi} & \\ & (\rho - \mu)e^{-2i\varphi} \end{pmatrix}, \\
U^T \begin{pmatrix} \alpha & a \\ a & \alpha \end{pmatrix} U &= \begin{pmatrix} (\alpha + a)e^{2i\varphi} & \\ & (\alpha - a)e^{-2i\varphi} \end{pmatrix}, \\
U &= \frac{1}{\sqrt{2}} e^{i\omega} \begin{pmatrix} e^{i\varphi} & -e^{-i\varphi} \\ e^{i\varphi} & e^{-i\varphi} \end{pmatrix}.
\end{aligned} \tag{107}$$

We can choose the particular case

$$U = U_0 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{108}$$

Call the initial basis

$$\begin{aligned}
\langle n_L | &= \begin{pmatrix} \langle \xi^\alpha | \\ \langle (\xi^\beta)^c | \end{pmatrix} \equiv \begin{pmatrix} \langle \xi^\alpha | \\ \langle -i(\eta^{\dot{\beta}})^\dagger | \end{pmatrix}, \quad \langle n_R | = \begin{pmatrix} \langle (\eta_{\dot{\alpha}})^c | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix} \equiv \begin{pmatrix} \langle -i(\xi_\alpha)^\dagger | \\ \langle \eta_{\dot{\beta}} | \end{pmatrix},
\end{aligned} \tag{109}$$

one has

$$\left(| \xi^\alpha \rangle \quad | (\xi^\beta)^c \rangle \quad | (\eta_{\dot{\gamma}})^c \rangle \quad | \eta_{\dot{\delta}} \rangle \right) = \left(| n_L \rangle \quad | n_R \rangle \right). \tag{110}$$

Define the new basis by

$$\begin{aligned}
\langle N_L | &= U_0^\dagger \langle n_L |, \quad \langle N_R | = U_0^\dagger \langle n_R |, \\
| N_L \rangle &= U_0 | n_L \rangle, \quad | N_R \rangle = U_0 | n_R \rangle.
\end{aligned} \tag{111}$$

One has explicitly

$$\begin{aligned}
\langle N_L | &= \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \xi^\alpha - i(\eta^{\dot{\alpha}})^\dagger | \\ \langle -\xi^\alpha - i(\eta^{\dot{\alpha}})^\dagger | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \xi^\alpha + (\xi^\alpha)^c | \\ \langle -\xi^\alpha + (\xi^\alpha)^c | \end{pmatrix}, \\
\langle N_R | &= \frac{1}{\sqrt{2}} \begin{pmatrix} \langle -i(\xi_\alpha)^\dagger + \eta_{\dot{\alpha}} | \\ \langle +i(\xi_\alpha)^\dagger + \eta_{\dot{\alpha}} | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \eta_{\dot{\alpha}} + (\eta_{\dot{\alpha}})^c | \\ \langle \eta_{\dot{\alpha}} - (\eta_{\dot{\alpha}})^c | \end{pmatrix},
\end{aligned} \tag{112}$$

and one can write

$$\langle N_L | = \begin{pmatrix} \langle \chi^\alpha | \\ \langle (-i)(\omega^{\dot{\beta}})^\dagger | \end{pmatrix}, \quad \langle N_R | = \begin{pmatrix} \langle (-i)(\chi_\alpha)^\dagger | \\ \langle \omega_{\dot{\beta}} | \end{pmatrix}. \tag{113}$$

In this new basis, the propagator writes (using (from (108)) $U_0^T U_0 = 1$)

$$\begin{aligned}
\Delta_{C+PCT}(p^2) &= \left(| N_L \rangle \quad | N_R \rangle \right) \\
&\left(\begin{array}{c|c} \begin{pmatrix} \alpha(p^2) + a(p^2) & \\ & \alpha(p^2) - a(p^2) \end{pmatrix} p_\mu \sigma_{\alpha\beta}^\mu & \begin{pmatrix} \rho(p^2) + \mu(p^2) & \\ & \rho(p^2) - \mu(p^2) \end{pmatrix} \delta_{\alpha\beta} \\ \hline \begin{pmatrix} \sigma(p^2) + m(p^2) & \\ & \sigma(p^2) - m(p^2) \end{pmatrix} \delta_{\alpha\beta} & \begin{pmatrix} \alpha(p^2) + a(p^2) & \\ & \alpha(p^2) - a(p^2) \end{pmatrix} p_\mu \overline{\sigma}^\mu_{\alpha\beta} \end{array} \right) \begin{pmatrix} \langle N_L | \\ \langle N_R | \end{pmatrix}.
\end{aligned} \tag{114}$$

Remember that $|u\rangle\langle v|$ corresponds, in our notation, to a propagator $\langle 0|\mathcal{T}u(x)v^\dagger(-x)|0\rangle$. One introduces the Majorana fermions (see subsection 3.6)

$$\begin{aligned} X_M^\pm &= \begin{pmatrix} \chi^\alpha \\ \pm(-i)(\chi_\alpha)^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^\alpha + (\xi^\alpha)^c \\ \pm(\eta_{\dot{\alpha}} + (\eta_{\dot{\alpha}})^c) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^\alpha - i(\eta^{\dot{\alpha}})^\dagger \\ \pm(\eta_{\dot{\alpha}} - i(\xi_\alpha)^\dagger) \end{pmatrix}, \\ \Omega_M^\pm &= \begin{pmatrix} \pm(-i)(\omega^{\dot{\beta}})^\dagger \\ \omega_{\dot{\beta}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm(-\xi^\beta + (\xi^\beta)^c) \\ \eta_{\dot{\beta}} - (\eta_{\dot{\beta}})^c \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm(-\xi^\beta - i(\eta^{\dot{\beta}})^\dagger) \\ \eta_{\dot{\beta}} + i(\xi_\beta)^\dagger \end{pmatrix}. \end{aligned} \quad (115)$$

5.6.1 Kinetic-like propagators

They can be rewritten

$$\begin{aligned} \int d^4x e^{ipx} \langle 0|\mathcal{T}\chi^\alpha(x)(\chi^\beta)^\dagger(-x)|0\rangle &= (\alpha(p^2) + a(p^2))p_\mu \sigma_{\alpha\beta}^\mu, \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}(\chi_\alpha)^\dagger(x)\chi_\beta(-x)|0\rangle &= (\alpha(p^2) + a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu, \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}(\omega^{\dot{\alpha}})^\dagger(x)\omega_{\dot{\beta}}(-x)|0\rangle &= (\alpha(p^2) - a(p^2))p_\mu \sigma_{\alpha\beta}^\mu, \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}\omega_{\dot{\alpha}}(x)(\omega_{\dot{\beta}})^\dagger(-x)|0\rangle &= (\alpha(p^2) - a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu, \end{aligned} \quad (116)$$

5.6.2 Mass-like propagators

They write

$$\begin{aligned} \int d^4x e^{ipx} \langle 0|\mathcal{T}\chi^\alpha(x)i\chi_\beta(-x)|0\rangle &= \delta_{\alpha\beta}(\rho(p^2) + \mu(p^2)), \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}(-i)(\chi_\alpha)^\dagger(x)(\chi^\beta)^\dagger(-x)|0\rangle &= \delta_{\alpha\beta}(\sigma(p^2) + m(p^2)), \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}(-i)(\omega^{\dot{\alpha}})^\dagger(x)(\omega_{\dot{\beta}})^\dagger(-x)|0\rangle &= \delta_{\alpha\beta}(\rho(p^2) - \mu(p^2)), \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}\omega_{\dot{\alpha}}(x)i\omega_{\dot{\beta}}(-x)|0\rangle &= \delta_{\alpha\beta}(\sigma(p^2) - m(p^2)). \end{aligned} \quad (117)$$

5.6.3 Conclusion

When C and PCT invariance holds, the fermion propagator decomposes into the propagators for the Majorana fermions X and Ω (115) (note that we have introduced below the $(\bar{})$ fields instead of the $()^\dagger$ fields, thus an extra γ^0 matrix)

$$\begin{aligned} \int d^4x e^{ipx} \langle 0|\mathcal{T}X_{M\alpha}^\pm(x)\overline{X_{M\beta}^\pm}(-x)|0\rangle &= \begin{pmatrix} (\rho(p^2) + \mu(p^2))\delta_{\alpha\beta} & (\alpha(p^2) + a(p^2))p_\mu \sigma_{\alpha\beta}^\mu \\ (\alpha(p^2) + a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu & (\sigma(p^2) + m(p^2))\delta_{\alpha\beta} \end{pmatrix}, \\ \int d^4x e^{ipx} \langle 0|\mathcal{T}\Omega_{M\alpha}^\pm(x)\overline{\Omega_{M\beta}^\pm}(-x)|0\rangle &= \begin{pmatrix} (\rho(p^2) - \mu(p^2))\delta_{\alpha\beta} & (\alpha(p^2) - a(p^2))p_\mu \sigma_{\alpha\beta}^\mu \\ (\alpha(p^2) - a(p^2))p_\mu \overline{\sigma}_{\alpha\beta}^\mu & (\sigma(p^2) - m(p^2))\delta_{\alpha\beta} \end{pmatrix}. \end{aligned} \quad (118)$$

(118) also writes

$$\begin{aligned}
& \frac{1}{2} \int d^4x e^{ipx} \left(\langle 0 | \mathcal{T} X_{M\alpha}^\pm(x) \overline{X_{M\beta}^\pm(-x)} | 0 \rangle + \langle 0 | \mathcal{T} \Omega_{M\alpha}^\pm(x) \overline{\Omega_{M\beta}^\pm(-x)} | 0 \rangle \right) \\
&= \begin{pmatrix} \rho(p^2) \delta_{\alpha\beta} & \alpha(p^2) p_\mu \sigma_{\alpha\beta}^\mu \\ \alpha(p^2) p_\mu \overline{\sigma}_{\alpha\beta}^\mu & \sigma(p^2) \delta_{\alpha\beta} \end{pmatrix}, \\
& \frac{1}{2} \int d^4x e^{ipx} \left(\langle 0 | \mathcal{T} X_{M\alpha}^\pm(x) \overline{X_{M\beta}^\pm(-x)} | 0 \rangle - \langle 0 | \mathcal{T} \Omega_{M\alpha}^\pm(x) \overline{\Omega_{M\beta}^\pm(-x)} | 0 \rangle \right) \\
&= \begin{pmatrix} \mu(p^2) \delta_{\alpha\beta} & a(p^2) p_\mu \sigma_{\alpha\beta}^\mu \\ a(p^2) p_\mu \overline{\sigma}_{\alpha\beta}^\mu & m(p^2) \delta_{\alpha\beta} \end{pmatrix}.
\end{aligned} \tag{119}$$

So, when $C + PCT$ invariance is realized, the most general fermion propagator is equivalent to two Majorana propagators.

The determinant of $\Delta(p^2)$ (114) is the products of the determinants of the matrices in the r.h.s. of (118); so, the poles of the two Majorana propagators in (118) are also poles of $\Delta(p^2)$, and the physical states (eigenstates of the propagator at its poles) are the Majorana fermions X and Ω .

5.7 Conditions for propagating Majorana eigenstates

We have shown in subsection 5.6 that, as expected since Majorana fermions are C eigenstates, a $C+PCT$ invariant propagator propagates Majorana fermions.

We now try to answer the reverse question *i.e.* which are the conditions on the propagator, in particular concerning discrete symmetries, for it to propagate Majorana fermions. This could look rather academic since we deal with one flavor and that it is “well known” that, in particular, no CP violating phase can occur in this case. So, we ask the reader to consider this section as a kind of intellectual exercise. In addition to being a preparation to the more complete study with several generations, it is also motivated by the fact that, in the propagator formalism (which differs from the one with a classical Lagrangian endowed with a mass matrix), even for one flavor, a fermion and its antifermions get mixed as soon as one allows all possible Lorentz invariant terms. That this peculiarity can *a priori* introduce a mixing angle between a particle and its antiparticle (like for neutral kaons) suggests that the situation may not be so trivial as naively expected. This section can also be considered as a test of the “common sense” statement that, since Majorana fermions are defined as C eigenstates, a propagator can only be expected to propagate Majorana fermions if it satisfies the constraints cast by C invariance. We shall indeed reach a conclusion close to this one in the following, with the only difference that CP symmetry also enters the game, for reasons that will be easy to understand (the general demonstration for a number of flavors greater than one, has been postponed to a further work).

5.7.1 General conditions for diagonalizing a PCT invariant propagator

We consider the most general PCT invariant propagator (98).

We are only concerned here with neutral fermions, for which diagonalizing each 2×2 sub-matrix of the propagator is meaningful: for charged fermions, this would mix in the same state fermions of different charges, which is impossible as soon as we assume that electric charge is conserved.

The two diagonal 2×2 sub-blocks involve differential operators, with one dotted and one undotted spinor index, factorized by simple functions of space-time. We will suppose that, inside each of these sub-blocks, the four differential operators are identical, such that their elements only differ by the functions of space-time. When we speak about diagonalizing these matrices, this concerns accordingly the space-time functions; then the differential operators follow naturally.

The mass-like sub-blocks are diagonal in spinor indices and involve only functions of space-time.

The propagator \mathcal{P} writes

$$\mathcal{P} = \left(\begin{array}{cc} |n_L\rangle & |n_R\rangle \end{array} \right) \left(\begin{array}{c|c} K_1 & M_1 \\ \hline M_2 & K_2 \end{array} \right) \left(\begin{array}{c} \langle n_L| \\ \langle n_R| \end{array} \right). \quad (120)$$

K_1, K_2, M_1 and M_2 have *a priori* no special properties, are not hermitian nor symmetric.

There always exist U_1 and U_2 , which have no reason to be unitary, such that

$$U_1^{-1}K_1U_1 = \Delta_1 \text{ diagonal}, \quad U_2^{-1}K_2U_2 = \Delta_2 \text{ diagonal}, \quad (121)$$

such that the propagator rewrites

$$\begin{aligned} \mathcal{P} &= \left(\begin{array}{cc} |n_L\rangle U_1 & |n_R\rangle U_2 \end{array} \right) \left(\begin{array}{c|c} \Delta_1 & U_1^{-1}M_1U_2 \\ \hline U_2^{-1}M_2U_1 & \Delta_2 \end{array} \right) \left(\begin{array}{c} U_1^{-1}\langle n_L| \\ U_2^{-1}\langle n_R| \end{array} \right) \\ &= \left(\begin{array}{cc} |\mathfrak{N}_L\rangle & |\mathfrak{N}_R\rangle \end{array} \right) \left(\begin{array}{c|c} \Delta_1 & U_1^{-1}M_1U_2 \\ \hline U_2^{-1}M_2U_1 & \Delta_2 \end{array} \right) \left(\begin{array}{c} \langle N_L| \\ \langle N_R| \end{array} \right), \\ \text{with} \quad &\langle N_L| = U_1^{-1}\langle n_L|, \quad \langle N_R| = U_2^{-1}\langle n_R|, \quad |\mathfrak{N}_L\rangle = |n_L\rangle U_1, \quad |\mathfrak{N}_R\rangle = |n_R\rangle U_2. \end{aligned} \quad (122)$$

The propagator can be diagonalized \Leftrightarrow

$$U_1^{-1}M_1U_2 = D_1 \text{ diagonal}, \quad U_2^{-1}M_2U_1 = D_2 \text{ diagonal}. \quad (123)$$

That $[D_1, D_2] = 0$ entails in particular

$$U_1^{-1}M_1M_2U_1 = D_1D_2 \text{ diagonal} = D_2D_1 = U_2^{-1}M_2M_1U_2, \quad (124)$$

which coincides with the commutation of M_1 and M_2 only when $U_1 = U_2$.

Since $[\Delta_1, D_1D_2] = 0 = [\Delta_2, D_1D_2]$, one also gets $U_1^{-1}[K_1, M_1M_2]U_1 = 0 = U_2^{-1}[K_2, M_2M_1]U_2$, which entails

$$[K_1, M_1M_2] = 0 = [K_2, M_2M_1]. \quad (125)$$

(121), (123), (124) and (125) are the conditions that K_1, K_2, M_1 and M_2 must satisfy for the propagator to be diagonalizable; they are must less stringent than the commutation of the four of them.

In practice: One supposes that M_1 and M_2 fulfill condition (125). To determine U_1 and U_2 , one can accordingly use indifferently (121) or (124): U_1 diagonalizes K_1 or M_1M_2 , U_2 diagonalizes K_2 or M_2M_1 . Supposing that (124) is satisfied, M_1M_2 and of M_2M_1 are constrained to have the same eigenvalues, which may give additional restrictions on M_1 and M_2 .

Once U_1 and U_2 are determined, call

$$\mathcal{M}_1 = U_1^{-1}M_1U_2, \quad \mathcal{M}_2 = U_2^{-1}M_2U_1. \quad (126)$$

(124) entails that, in particular, \mathcal{M}_1 and \mathcal{M}_2 must commute. Since U_1 diagonalizes M_1M_2 and U_2 diagonalizes M_2M_1 , $\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_2\mathcal{M}_1$ are diagonal.

Write $\mathcal{M}_1 = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix}$ and $\mathcal{M}_2 = \begin{pmatrix} \mathfrak{p} & \mathfrak{q} \\ \mathfrak{r} & \mathfrak{s} \end{pmatrix}$; by direct inspection, one finds that the two products

$\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_2\mathcal{M}_1$ are diagonal either if \mathcal{M}_1 and \mathcal{M}_2 are diagonal, or if $\mathcal{M}_2 = t \begin{pmatrix} \mathfrak{d} & -\mathfrak{b} \\ -\mathfrak{c} & \mathfrak{a} \end{pmatrix}$, that

is, is proportional to \mathcal{M}_1^{-1} ; in this last case, $\mathcal{M}_1\mathcal{M}_2 = \mathcal{M}_2\mathcal{M}_1$ is proportional to the unit matrix, which means that the eigenvalues of M_1M_2 are all identical (and so are the eigenvalues of M_2M_1).

We are looking for more: the conditions that must satisfy M_1 and M_2 for \mathcal{M}_1 and \mathcal{M}_2 to be separately diagonal. We attempt to find them by putting the additional restriction that the eigenstates are Majorana fermions.

5.7.2 Condition for propagating Majorana fermions

A necessary (but not sufficient) condition for the propagating states to be Majorana is that, by some change of basis, the propagator can be cast in the form

$$\Delta_{Maj}(p^2) = \left(\begin{array}{c|c} \left(\begin{array}{cc} a_1(p^2) & \\ & b_1(p^2) \end{array} \right) p_\mu \sigma^\mu_{\alpha\beta} & \left(\begin{array}{cc} m_1(p^2) & \\ & \mu_1(p^2) \end{array} \right) \delta_{\alpha\beta} \\ \hline \left(\begin{array}{cc} m_2(p^2) & \\ & \mu_2(p^2) \end{array} \right) \delta_{\alpha\beta} & \left(\begin{array}{cc} a_2(p^2) & \\ & b_2(p^2) \end{array} \right) p_\mu \overline{\sigma}^\mu_{\alpha\beta} \end{array} \right), \quad (127)$$

with four diagonal 2×2 sub-blocks. Indeed, one can then decompose the propagator into two 4×4 propagators (in a shortened notation) $\begin{pmatrix} a_1 & m_1 \\ m_2 & a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 & \mu_1 \\ \mu_2 & b_2 \end{pmatrix}$, and the Majorana fermions (see subsection 3.6) are eventually, respectively, composed with the first components of n_L and n_R , and with the second components of the same set. So, in particular, both kinetic-like and mass-like terms, should be diagonalizable simultaneously²⁵. We note

$$U_1^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U_2^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & \delta_1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} d_2 & 0 \\ 0 & \delta_2 \end{pmatrix}. \quad (128)$$

One has

$$\begin{aligned} \langle N_L | &= \left(\begin{array}{l} a < \xi^\alpha \mid + b < (-i)(\eta^{\dot{\alpha}})^* \mid \\ c < \xi^\alpha \mid + d < (-i)(\eta^{\dot{\alpha}})^* \mid \end{array} \right), \\ \langle N_R | &= \left(\begin{array}{l} p < (-i)\xi_\alpha^* \mid + q < \eta_{\dot{\alpha}} \mid \\ r < (-i)\xi_\alpha^* \mid + s < \eta_{\dot{\alpha}} \mid \end{array} \right), \\ | \mathfrak{N}_L \rangle &= \frac{1}{ad - bc} \left(\begin{array}{l} d \mid \xi^\alpha > -c \mid (-i)(\eta^{\dot{\alpha}})^* > \\ -b \mid \xi^\alpha > +a \mid (-i)(\eta^{\dot{\alpha}})^* > \end{array} \right), \\ | \mathfrak{N}_R \rangle &= \frac{1}{ps - qr} \left(\begin{array}{l} s \mid (-i)\xi_\alpha^* > -r \mid \eta_{\dot{\alpha}} > \\ -q \mid (-i)\xi_\alpha^* > +p \mid \eta_{\dot{\alpha}} > \end{array} \right), \end{aligned} \quad (129)$$

and the question is whether the propagator $\langle 0 \mid \mathcal{T} \begin{pmatrix} N_L(x) \\ N_R(x) \end{pmatrix} \begin{pmatrix} \mathfrak{N}_L(-x) & \mathfrak{N}_R(-x) \end{pmatrix}^\dagger \mid 0 \rangle$ can be identified with that of a Majorana fermion and its antifermion (that is, itself). Eq. (129) yields in particular the four mass-like propagators

$$\begin{aligned} \langle 0 \mid \mathcal{T} \left(d\xi^\alpha + ic(\eta^{\dot{\alpha}})^\dagger \right) (x) \left(ip^* \xi_\beta + q^*(\eta_{\dot{\beta}})^\dagger \right) (-x) \mid 0 \rangle &= (ad - bc)d_1(x)\delta_{\alpha\beta}, \quad (a) \\ \langle 0 \mid \mathcal{T} \left(-b\xi^\alpha - ia(\eta^{\dot{\alpha}})^\dagger \right) (x) \left(ir^* \xi_\beta + s^*(\eta_{\dot{\beta}})^\dagger \right) (-x) \mid 0 \rangle &= (ad - bc)\delta_1(x)\delta_{\alpha\beta}, \quad (b) \\ \langle 0 \mid \mathcal{T} \left(-is(\xi_\alpha)^\dagger - r\eta_{\dot{\alpha}} \right) (x) \left(a^*(\xi^\beta)^\dagger + ib^*\eta^{\dot{\beta}} \right) (-x) \mid 0 \rangle &= (ps - qr)d_2(x)\delta_{\alpha\beta}, \quad (c) \\ \langle 0 \mid \mathcal{T} \left(iq(\xi_\alpha)^\dagger + p\eta_{\dot{\alpha}} \right) (x) \left(c^*(\xi^\beta)^\dagger + id^*\eta^{\dot{\beta}} \right) (-x) \mid 0 \rangle &= (ps - qr)\delta_2(x)\delta_{\alpha\beta}, \quad (d) \end{aligned} \quad (130)$$

which must be the only four non-vanishing such propagators since $U_1^{-1}M_1U_2$ and $U_2^{-1}M_2U_1$ must be diagonal. We have to identify them with typical mass-like Majorana propagators. For that purpose, we

²⁵Imposing commutation relations between all 2×2 sub-blocks of the propagator is excessive.

have a priori to introduce two Majorana fermions; $X_M^\pm = \begin{pmatrix} \zeta^\alpha \\ \pm(-i)(\zeta_\alpha)^* \end{pmatrix}$, associated, together with its antifermion, to (N_L, N_R) , and $Y_M^\pm = \begin{pmatrix} \chi^\beta \\ \pm(-i)(\chi_\beta)^* \end{pmatrix}$, associated, together with its antifermion, to $(\mathfrak{N}_L, \mathfrak{N}_R)$. An $X - Y$ propagator²⁶ reads (we go to the $(\bar{})$ fields, which introduces an extra γ^0 ; this has in particular for consequence that “mass-like” propagators now appear on the diagonal)

$$\langle 0 | \mathcal{T} X_M(x) \overline{Y}_M(-x) | 0 \rangle = \begin{pmatrix} \langle 0 | \mathcal{T} \zeta^\alpha(x) (\pm i) \chi_\beta(-x) | 0 \rangle & \langle 0 | \mathcal{T} \zeta^\alpha(x) (\chi^\beta)^\dagger(-x) | 0 \rangle \\ \langle 0 | \mathcal{T} (\zeta_\alpha)^\dagger(x) \chi_\beta(-x) | 0 \rangle & \langle 0 | \mathcal{T} (\mp i) (\zeta_\alpha)^\dagger(x) (\chi^\beta)^\dagger(-x) | 0 \rangle \end{pmatrix}. \quad (131)$$

The four lines of (130) correspond to two mass-like $X - Y$ propagators only if one can associate them into two pairs, such that each pair has the same structure as the diagonal terms of (131). There are accordingly two possibilities: pairing (a) with (c) and (b) with (d), or (a) with (d) and (b) with (c).

* The first possibility requires (κ and λ are proportionality constants) $p = i\lambda a^*$, $q = i\lambda b^*$, $r = -i\kappa c^*$, $s = -i\kappa d^*$, such that

$$U_2^{-1} = i \begin{pmatrix} \lambda a^* & \lambda b^* \\ -\kappa c^* & -\kappa d^* \end{pmatrix}. \quad (132)$$

* The second possibility requires $p = i\rho c^*$, $q = i\rho d^*$, $r = i\theta a^*$, $s = i\theta b^*$ such that

$$U_2^{-1} = i \begin{pmatrix} \rho c^* & \rho d^* \\ \theta a^* & \theta b^* \end{pmatrix}. \quad (133)$$

From now onwards, we furthermore request that a single Majorana fermion propagates in the sense that only \mathcal{T} -products of the type $\langle 0 | \mathcal{T} X^\alpha(x) X_\alpha(-x) | 0 \rangle$ occur, which associates $|\mathcal{N}_L\rangle = |X^\alpha\rangle$ and $\langle N_R| = \langle X_\alpha^*|$. The only possibility is that the coefficients of $|\mathcal{N}_L\rangle$ and $|\mathcal{N}_R\rangle$ in (129) be proportional, and so be the ones of $|\mathcal{N}_R\rangle$ and $|N_L\rangle$ (the two sets of conditions are the same); this gives the supplementary conditions (σ and β are two other proportionality constants) $p = i\sigma d^*$, $q = -i\sigma c^*$, $r = -i\gamma b^*$, $s = i\gamma a^*$, such that

$$U_2^{-1} = i \begin{pmatrix} \sigma d^* & -\sigma c^* \\ -\gamma b^* & \gamma a^* \end{pmatrix}. \quad (134)$$

* First possibility (U_2^{-1} is given by (132) above).

Compatibility between (132) and (134) requires $\frac{q}{p} = \frac{b^*}{a^*} = -\frac{c^*}{d^*} = -\frac{r}{s} = \omega^*$ such that we end up with

$$U_1^{-1} = \begin{pmatrix} a & \omega a \\ -\omega d & d \end{pmatrix}, \quad U_2^{-1} = \begin{pmatrix} p & \omega^* p \\ -\omega^* s & s \end{pmatrix} = \begin{pmatrix} \lambda a^* & \lambda \omega^* a^* \\ \kappa \omega^* d^* & -\kappa d^* \end{pmatrix}. \quad (135)$$

We look for PCT invariant $M_1 = \begin{pmatrix} m_{L1}(x) & \mu_1(x) \\ \mu_1(x) & m_{R1}(x) \end{pmatrix}$ and $M_2 = \begin{pmatrix} m_{L2}(x) & \mu_2(x) \\ \mu_2(x) & m_{R2}(x) \end{pmatrix}$ (see (98)) and their diagonalization according to (123) and (128) by U_1 and U_2 given by (135) and satisfying (125).

²⁶We allow here $X \neq Y$, but will then become more restrictive by requesting $X = Y$, which better corresponds to the intuitive picture of propagating a definite Majorana fermion.

The equations (121) of diagonalization for the kinetic-like terms $K_1 = \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$ and $K_2 = \begin{pmatrix} \alpha & v \\ u & \beta \end{pmatrix}$ (see (98)) yield, for the vanishing of the non-diagonal terms, the conditions

$$\begin{aligned} u - \omega^2 v &= \omega(\alpha - \beta), \\ v - \omega^2 u &= \omega(\alpha - \beta), \\ v - \omega^{*2} u &= \omega^*(\alpha - \beta), \\ u - \omega^{*2} v &= \omega^*(\alpha - \beta). \end{aligned} \quad (136)$$

Likewise, the diagonalization equations (123) for the mass-like terms yield

$$\begin{aligned} \omega^* m_{L1} - \omega m_{R1} &= \mu_1(1 - |\omega|^2), \\ \omega m_{L1} - \omega^* m_{R1} &= \mu_1(1 - |\omega|^2), \\ \omega^* m_{L2} - \omega m_{R2} &= \mu_2(1 - |\omega|^2), \\ \omega m_{L2} - \omega^* m_{R2} &= \mu_2(1 - |\omega|^2). \end{aligned} \quad (137)$$

First, we eliminate the trivial case $\omega = 1$ which brings back to a C invariant propagator.

Subtracting the first or the last two equations of (136) yields $u = v$. One then gets $\alpha - \beta = u \frac{1-\omega^2}{\omega} = u \frac{1-\omega^{*2}}{\omega^*}$, such that ω must be real.

Subtracting the first two equations of (137) also shows that ω must be real as soon as one supposes $m_{L1} + m_{R1} \neq 0$, which we do. Then, one gets $\frac{\mu_1}{m_{L1}-m_{R1}} = \frac{\omega}{1-\omega^2} = \frac{\mu_2}{m_{L2}-m_{R2}}$. Gathering the results from (136) and (137) leads accordingly to

$$\begin{aligned} K_1 &= u \begin{pmatrix} \alpha & (\alpha - \beta) \frac{\omega}{1 - \omega^2} \\ (\alpha - \beta) \frac{\omega}{1 - \omega^2} & \beta \end{pmatrix} = K_2, \\ M_1 &= \begin{pmatrix} m_{L1} & (m_{L1} - m_{R1}) \frac{\omega}{1 - \omega^2} \\ (m_{L1} - m_{R1}) \frac{\omega}{1 - \omega^2} & m_{R1} \end{pmatrix}, \\ M_2 &= \begin{pmatrix} m_{L2} & (m_{L2} - m_{R2}) \frac{\omega}{1 - \omega^2} \\ (m_{L2} - m_{R2}) \frac{\omega}{1 - \omega^2} & m_{R2} \end{pmatrix}, \end{aligned} \quad (138)$$

and we shall hereafter write $\omega = \tan \vartheta$. The four real symmetric matrices $K_1 = K_2, M_1, M_2$ can be simultaneously diagonalized by the same rotation matrix $U(\vartheta)$ of angle ϑ . After diagonalization, the propagator writes

$$\begin{aligned} \Delta &= \left(\begin{array}{c|c} |n_L\rangle U & |n_R\rangle U \end{array} \right) \left(\begin{array}{c|c} \delta_+ & \mu_{1+} \\ \hline \delta_- & \mu_{1-} \\ \hline \mu_{2+} & \delta_+ \\ \hline \mu_{2-} & \delta_- \end{array} \right) \left(\begin{array}{c} U^T \langle n_L| \\ U^T \langle n_R| \end{array} \right), \\ \text{with} \quad \delta_{\pm} &= \frac{1}{2} \left(\alpha + \beta \pm \frac{\alpha - \beta}{\cos 2\vartheta} \right), \quad \mu_{1,2,\pm} = \frac{1}{2} \left(m_{L1,2} + m_{R1,2} \pm \frac{m_{L1,2} - m_{R1,2}}{\cos 2\vartheta} \right) \end{aligned} \quad (139)$$

To propagate a Majorana fermion, the condition $\mu_{1+} = \mu_{2+}$ should furthermore be fulfilled. This requires, for arbitrary ϑ , $m_{R1} = m_{R2}, m_{L1} = m_{L2}$ (and thus $\mu_1 = \mu_2$). This corresponds to a propagator

(before diagonalization)

$$\Delta = \left(\begin{array}{c|c} |n_L\rangle & |n_R\rangle \end{array} \right) \left(\begin{array}{cc|cc} \alpha & u & m_L & \mu \\ u & \beta & \mu & m_R \\ \hline m_L & \mu & \alpha & u \\ \mu & m_R & u & \beta \end{array} \right) \left(\begin{array}{c} \langle n_L| \\ \langle n_R| \end{array} \right), \quad \frac{u}{\alpha - \beta} = \frac{\mu}{m_L - m_R}, \quad (140)$$

that is, a CP invariant propagator (see (105)) (the C invariant case corresponds to $\omega = 1$ (see (100)), which has been treated previously). The propagating Majorana fermion are

$$\psi_M = \begin{pmatrix} \cos \vartheta \xi^\alpha - \sin \vartheta (-i(\eta^{\dot{\alpha}})^*) \\ \cos \vartheta (-i(\xi_\gamma)^*) - \sin \vartheta \eta_{\dot{\gamma}} \end{pmatrix} \text{ and } \chi_M = \begin{pmatrix} \sin \vartheta \xi^\alpha + \cos \vartheta (-i(\eta^{\dot{\beta}})^*) \\ \sin \vartheta (-i(\xi_\gamma)^*) + \cos \vartheta \eta_{\dot{\gamma}} \end{pmatrix}.$$

* Second possibility (U_2^{-1} is given by (133) above). Equating (134), (133) and the expression for U_2^{-1} in (128), one gets $q/p = d^*/c^* = -c^*/d^*$, $s/r = b^*/a^* = -a^*/b^*$, which gives $d = \pm ic$, $b = \pm ia$ and thus

$$U_1^{-1} = \begin{pmatrix} a & \pm ia \\ c & \pm ic \end{pmatrix}, \quad U_2^{-1} = i \begin{pmatrix} \rho c^* & \mp i \rho c^* \\ \pm i \gamma a^* & \gamma a^* \end{pmatrix}. \quad (141)$$

The diagonalization equations (123) for the mass-like terms yield, for the vanishing of the non-diagonal terms, the conditions

$$\begin{aligned} m_{L1} &= -m_{R1}, \\ m_{L2} &= -m_{R2}. \end{aligned} \quad (142)$$

The equations (121) of diagonalization for the kinetic-like terms yield the conditions

$$\begin{aligned} u + v &= \pm i(\alpha - \beta), \\ u + v &= \pm i(\beta - \alpha), \end{aligned} \quad (143)$$

which require $v = -u$, $\beta = \alpha$.

So, the kinetic and mass-like propagators write

$$\begin{aligned} K_1 &= \begin{pmatrix} \alpha & u \\ -u & \alpha \end{pmatrix}, & K_2 &= \begin{pmatrix} \alpha & -u \\ u & \alpha \end{pmatrix}, \\ M_1 &= \begin{pmatrix} m_1 & \mu_1 \\ \mu_1 & -m_1 \end{pmatrix}, & M_2 &= \begin{pmatrix} m_2 & \mu_2 \\ \mu_2 & -m_2 \end{pmatrix}. \end{aligned} \quad (144)$$

K_1 and K_2 , which commute, can be diagonalized simultaneously by a single matrix U . The conditions (125) $[K_1, M_1 M_2] = 0 = [K_2, M_2 M_1]$ require $m_1/m_2 = \mu_1/\mu_2$, such that $M_2 = \chi M_1$. Since $U_1 = U = U_2$, the diagonalization equations (123) for the mass-like propagators rewrite $U^{-1} M_1 U = D_1$, $U^{-1} M_2 U = \chi D_1$, such that the set of four matrices K_1, K_2, M_1, M_2 must commute, which requires $u = 0$. The kinetic-like propagators are thus “standard”, *i.e.* proportional to the unit matrix. Before diagonalization, the propagator writes

$$\Delta = \left(\begin{array}{c|c} |n_L\rangle & |n_R\rangle \end{array} \right) \left(\begin{array}{cc|cc} \alpha & & m_1 & \mu_1 \\ & \alpha & \mu_1 & -m_1 \\ \hline \chi m_1 & \chi \mu_1 & \alpha & \\ \chi \mu_1 & -\chi m_1 & & \alpha \end{array} \right) \left(\begin{array}{c} \langle n_L| \\ \langle n_R| \end{array} \right), \quad (145)$$

and, after diagonalization,

$$\Delta = \left(\begin{array}{c|c} |n_L\rangle & |n_R\rangle \\ \hline \end{array} U \right) \left(\begin{array}{c|c} \alpha & \mu \\ \hline \chi\mu & \alpha \\ \hline -\chi\mu & \alpha \end{array} \right) \left(\begin{array}{c} U^T < n_L \\ U^T < n_R \end{array} \right),$$

with $\mu = \sqrt{m_1^2 + \mu_1^2}$. (146)

It can propagate Majorana fermions only if $\chi = 1$, such that $M_1 = M_2$. Then, (145) is a special kind of PC invariant propagator (see (105)), which becomes C invariant only when $m_1 = 0$. The two Majorana fermions have masses $\pm\mu/\alpha$. They are $\psi_M = \begin{pmatrix} \cos\vartheta\xi^\alpha - \sin\vartheta(-i(\eta^\alpha)^*) \\ \cos\vartheta(-i(\xi_\gamma)^*) - \sin\vartheta\eta_{\dot{\gamma}} \end{pmatrix}$ and $\chi_M = \begin{pmatrix} \sin\vartheta\xi^\alpha + \cos\vartheta(-i(\eta^\beta)^*) \\ \sin\vartheta(-i(\xi_\gamma)^*) + \cos\vartheta\eta_{\dot{\gamma}} \end{pmatrix}$, with $\tan 2\vartheta = \mu_1/m_1$.

5.7.3 Conclusion

For one flavor (particle + antiparticle), a necessary condition for the eigenstates of the propagator to be Majorana is either that this propagator (supposed to satisfy the constraints cast by PCT invariance) satisfies the constraints cast by C invariance (which corresponds to $\omega = 1$) or by CP invariance²⁷. So, reciprocally, if the most general PCT invariant propagator for one flavor does not satisfy the constraints cast by C nor the ones cast by CP , its eigenstates cannot be Majorana.

6 General conclusion

In this work, we have extended the propagator approach [3] [4] [1] to coupled fermionic systems. It is motivated, in particular, by the ambiguities that unavoidably occur when dealing with a classical fermionic Lagrangian endowed with a mass matrix. The goal of this formalism is, in particular, to determine at which condition the propagating neutral fermions, defined as the eigenstates, at the poles, of their full propagator, are Majorana. Due to the intricacies of this approach, we presently limited ourselves to the simplest case of a single fermion and its antifermion. Since Lorentz invariance allows that they get coupled (as long as it is not forbidden by electric charge conservation), one can expect properties similar to the ones of the neutral kaons system. In this simple case, we have proved what is suggested by common sense, *i.e.* that the propagating fermions can only be Majorana if their propagator satisfies the constraints cast by C (or CP) invariance.

The generalization to several flavors will be the object of a subsequent work, with, in particular, the persistent goal of unraveling the nature of neutrinos.

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²⁷This is linked to the property of Majorana fermions to have CP parity $= \pm i$ (see subsections 3.3 and 3.6). The two corresponding $(\pm i\gamma^0)$ factors cancel in the \mathcal{T} -product of their propagator, which makes it CP invariant. This explains why not only C invariant, but also CP invariant propagators can propagate Majorana fermions

A Notations: spinors

A.1 Weyl spinors

We adopt the notations of [12], with undotted and dotted indices.

Undotted spinors, contravariant ξ^α or covariant ξ_α can be also called left spinors. Dotted spinors, covariant $\eta_{\dot{\alpha}}$ or contravariant $\eta^{\dot{\alpha}}$ can then be identified as right spinors. They are 2-components complex spinors. The 2-valued spinor indices are not explicitly written.

By an arbitrary transformation of the proper Lorentz group

$$\alpha\delta - \beta\gamma = 1, \quad (147)$$

they transform by

$$\begin{aligned} \xi^{1'} &= \alpha\xi^1 + \beta\xi^2, \\ \xi^{2'} &= \gamma\xi^1 + \delta\xi^2, \\ \eta^{\dot{1}'} &= \alpha^*\eta^{\dot{1}} + \beta^*\eta^{\dot{2}}, \\ \eta^{\dot{2}'} &= \gamma^*\eta^{\dot{1}} + \delta^*\eta^{\dot{2}}. \end{aligned} \quad (148)$$

To raise or lower spinor indices, one has to use the metric of $SL(2, C)$

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_{\alpha\beta}^2; \quad g^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i(\sigma^2)_{\alpha\beta}, \quad (149)$$

and the same for dotted indices. The σ^2 matrix will always be represented with indices down.

$$\xi_\alpha = g_{\alpha\beta}\xi^\beta = i\sigma_{\alpha\beta}^2\xi^\beta, \eta_{\dot{\alpha}} = g_{\dot{\alpha}\dot{\beta}}\eta^{\dot{\beta}} = -i\sigma_{\dot{\alpha}\dot{\beta}}^2\eta^{\dot{\beta}}. \quad (150)$$

One has

$$\xi.\zeta = \xi^\alpha\zeta_\alpha = \xi^1\zeta^2 - \xi^2\zeta^1 = -\xi_\alpha\zeta^\alpha \text{ invariant}. \quad (151)$$

By definition, $\eta_{\dot{\alpha}} \sim \xi_\alpha^*$ (transforms as);

$$\eta_{\dot{\alpha}} \sim (g_{\alpha\beta}\xi^\beta)^* = g_{\alpha\beta}(\xi^\beta)^* = i\sigma_{\alpha\beta}^2\xi^{\beta*} : \quad (152)$$

a right-handed Weyl spinor and the complex conjugate of a left-handed Weyl spinor transform alike by Lorentz; likewise, a left-handed spinor transforms like the complex conjugate of a right-handed spinor.

A Dirac (bi-)spinor is

$$\xi_D = \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{pmatrix}. \quad (153)$$

A.2 Pauli and Dirac matrices

Since we work with Weyl fermions, we naturally choose the Weyl representation.

Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (154)$$

γ matrices

$$\gamma^0 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right), \gamma^i = \left(\begin{array}{cc} 0 & -\sigma^i \\ \sigma^i & 0 \end{array} \right), \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right), \quad (155)$$

and one notes

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) = \gamma^0 \left(\begin{array}{cc} \sigma^\mu & 0 \\ 0 & \overline{\sigma^\mu} \end{array} \right), \quad (156)$$

with

$$\sigma^\mu = (\sigma^0, \vec{\sigma}), \quad \overline{\sigma^\mu} = (\sigma^0, -\vec{\sigma}), \quad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \quad (157)$$

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0, (\gamma^5)^\dagger = \gamma^5, (\gamma^{1,2,3})^\dagger = -\gamma^{1,2,3}, \\ (\gamma^0)^* &= \gamma^0, (\gamma^5)^* = \gamma^5, (\gamma^{1,3})^* = \gamma^{1,3}, (\gamma^2)^* = -\gamma^2, \\ (\gamma^0)^2 &= 1, (\gamma^5)^2 = 1, (\gamma^{1,2,3})^2 = -1, \\ \gamma^0(\gamma^0)^\dagger &= 1, \gamma^5(\gamma^5)^\dagger = 1, \gamma^{1,2,3}(\gamma^{1,2,3})^\dagger = 1. \end{aligned} \quad (158)$$

One has

$$(\sigma^0)^2 = 1 = (\sigma^i)^2, \{\sigma^i, \sigma^j\} = 2\delta^{ij}. \quad (159)$$

One has the relation

$$\sigma_{\beta\delta}^2 \sigma_{\alpha\gamma}^2 = \delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\delta\gamma}, \quad (160)$$

and the following one is very useful

$$\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*, \quad \sigma^2 \sigma^0 \sigma^2 = \sigma^0 \Rightarrow \sigma^2 \sigma^\mu \sigma^2 = (\sigma^0, -\vec{\sigma}^*) = \overline{\sigma^\mu}^*. \quad (161)$$

As far as kinetic terms are concerned,

$$\gamma^0 \gamma^\mu p_\mu = (\gamma^0)^2 p_\mu \left(\begin{array}{cc} \sigma^\mu & 0 \\ 0 & \overline{\sigma^\mu} \end{array} \right) = \left(\begin{array}{cc} p^0 - \vec{p} \cdot \vec{\sigma} & 0 \\ 0 & p^0 + \vec{p} \cdot \vec{\sigma} \end{array} \right). \quad (162)$$

B The adjoint of an antilinear operator

Following Weinberg [11], let us show that the adjoint of an antilinear operator (see (5) for the definition) \mathcal{A} cannot be defined by $\langle \mathcal{A}\psi | \chi \rangle = \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ ²⁸. Indeed, suppose that we can take the usual definition above, and let c be a c-number; using the antilinearity of \mathcal{A} one gets $\langle \mathcal{A}(c\psi) | \chi \rangle = \langle c^*(\mathcal{A}\psi) | \chi \rangle = c \langle \mathcal{A}\psi | \chi \rangle = c \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ is linear in ψ .

But one has also $\langle \mathcal{A}(c\psi) | \chi \rangle = \langle c\psi | \mathcal{A}^\dagger | \chi \rangle = \langle \psi | c^* \mathcal{A}^\dagger | \chi \rangle = c^* \langle \psi | \mathcal{A}^\dagger | \chi \rangle$ is antilinear in ψ , which is incompatible with the result above. So, the two expressions cannot be identical and $\langle \mathcal{A}\psi | \chi \rangle \neq \langle \psi | \mathcal{A}^\dagger | \chi \rangle$.

Weinberg ([11] p.51) defines the adjoint by²⁹

$$\langle \psi | \mathcal{A}^\dagger | \chi \rangle \equiv \langle \psi | \mathcal{A}^\dagger | \chi \rangle = \langle \mathcal{A}\psi | \chi \rangle^* = \langle \chi | \mathcal{A} \psi \rangle \equiv \langle \chi | \mathcal{A} | \psi \rangle \quad (164)$$

²⁸This changes nothing to our demonstrations.

²⁹So defined, taking $\psi = \chi$, the adjoint satisfies $\langle \psi | \mathcal{A} | \psi \rangle = \langle \psi | \mathcal{A}^\dagger | \psi \rangle$. This entails in particular that, for a antiunitary operator

$$\langle \psi | \mathcal{A}^\dagger | \psi \rangle \neq \langle \psi | \mathcal{A} | \psi \rangle, \quad (163)$$

unless what happens for antiunitary operators (otherwise the matrix element $\langle \psi | \mathcal{A} | \psi \rangle$ of any antiunitary operator could only be real, which is nonsense).

Then, even for an antilinear and antiunitary operator one has ³⁰

$$\mathcal{A}^\dagger \mathcal{A} = 1. \quad (165)$$

Indeed, $\langle \psi | \mathcal{A}^\dagger \mathcal{A} | \chi \rangle = \langle \psi | \mathcal{A}^\dagger | \mathcal{A} \chi \rangle \stackrel{(164)}{=} \langle \mathcal{A} \chi | \mathcal{A} | \psi \rangle = \langle \mathcal{A} \chi | \mathcal{A} \psi \rangle \stackrel{\text{antiunitarity}}{=} \langle \psi | \chi \rangle$.

By a similar argument, and because \mathcal{A}^\dagger is also antiunitary, one shows that one can also take $\mathcal{A} \mathcal{A}^\dagger = 1$.

So, both linear unitary \mathcal{U} and antilinear antiunitary \mathcal{A} operators satisfy

$$\mathcal{U} \mathcal{U}^\dagger = 1 = \mathcal{U}^\dagger \mathcal{U}, \quad \mathcal{A} \mathcal{A}^\dagger = 1 = \mathcal{A}^\dagger \mathcal{A}. \quad (166)$$

C Classical versus quantum Lagrangian; complex versus hermitian conjugation

In most literature, a fermionic Lagrangian (specially for neutrinos), is completed by its complex conjugate. This is because, at the classical level, a Lagrangian is a scalar and the fields in there are classical fields, not operators.

However, when fields are quantized, they become operators, so does the Lagrangian which is a sum of (local) products of fields, such that, in this case, the complex conjugate should be replaced by the hermitian conjugate.

Consider for example two Dirac fermions $\chi = \begin{pmatrix} \xi^\alpha \\ \eta_{\dot{\beta}} \end{pmatrix}$ and $\psi = \begin{pmatrix} \varphi^\alpha \\ \omega_{\dot{\beta}} \end{pmatrix}$; a typical mass term in a classical Lagrangian reads $\overline{\chi}_L \psi_R = (\xi^\alpha)^* \omega_{\dot{\alpha}} = \xi^{\dot{\alpha}} \omega_{\dot{\alpha}} = -\omega_{\dot{\alpha}} \xi^{\dot{\alpha}} = \omega^{\dot{\alpha}} \xi_{\dot{\alpha}}$, where we have supposed that ξ and ω anticommute; its complex conjugate reads then $(\overline{\chi}_L \psi_R)^* = \omega^\alpha \xi_\alpha = (\omega^{\dot{\alpha}})^* \xi_{\dot{\alpha}}$.

If we now consider operators $(\overline{\chi}_L \psi_R) = [\xi^\alpha]^\dagger [\omega_{\dot{\alpha}}] = [\chi_L]^\dagger [\psi_R]$, and its hermitian conjugate is $[\omega_{\dot{\alpha}}]^\dagger [\xi^\alpha] = [\omega_{\dot{\alpha}}^*] [\xi^\alpha]$. Since $([\chi_L]^\dagger [\psi_R])^\dagger = [\psi_R]^\dagger [\chi_L]$, it only ‘coincides’ with the classical complex conjugate if we adopt the convention

$$\psi_R^\dagger \chi_L = (\omega^{\dot{\beta}})^* \xi_{\dot{\beta}}, \quad (167)$$

where one has raised the index of ω and lowered the one of ξ . We will hereafter adopt (167).

D On the use of effective expressions for the P , C and T operators when acting on a Dirac fermion

In the body of this paper we have chosen to work with fundamental Weyl fermions ξ^α and $\eta_{\dot{\alpha}}$. In order to determine how the discrete symmetries P , C and T act on them, we started by their action on Dirac fermions in terms of γ matrices, from which, then, we deduced how each component transforms.

However, one must be very cautious concerning the way P , C and T act in terms of Dirac γ matrices; this notation can indeed easily cause confusion and induce into error, as we show below. It can be specially misleading when calculating the action of various products of these three transformations and only an extreme care can prevent from going astray. This is why, in manipulating these symmetry operators, we take as a general principle to strictly use their action on Weyl fermions, together with the knowledge of their linearity or antilinearity.

Since, nevertheless, the Dirac formalism is of very common use among physicists, we also give in the following the correct rules for manipulating, in this framework, discrete transformations and their products.

Let K be a transformation acting as follows on a Dirac fermion ψ_D : $K \cdot \psi_D = U_K \psi_D^{(*)}$, where U_K is a matrix which is in general unitary. In the case of the usual transformations P , C and T , U_K may be

³⁰This is in contradiction with [13].

expressed in terms of γ matrices. One must however keep in mind that this does not provide a complete characterization of the corresponding transformation, but only an effective one that must be handled with extreme care. It can indeed be misleading, specially if one relies on “intuition” to infer from this expression the linearity or antilinearity of the transformation under consideration. This is what we showed in subsection 3.2 concerning charge conjugation. Indeed, $P \cdot \psi_D = i\gamma^0 \psi_D$ and P is *linear* (unitary); $C \cdot \psi_D = \gamma^2 \psi_D^*$ and C is *linear* (unitary); $T \cdot \psi_D = i\gamma^3 \gamma^1 \psi_D^*$ and T is *antilinear* (antiunitary); $PCT \cdot \psi_D = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi_D$ and PCT is *antilinear* (antiunitary).

To illustrate this, let us investigate three *a priori* possible ways of computing the action of PCT , and compare them with the correct result, obtained by applying directly to Weyl fermions the three transformations successively (taking into account the linear or antilinear character of operators):

- * the crudest way consists in basically multiplying the U_K 's, without considering any action on a spinor (hence neglecting any consideration concerning complex conjugation);
- * the second one [12], that we call “Landau” uses as a rule the composition of the symmetry actions on a Dirac spinor;
- * the third one consists of making use of the linearity/antilinearity of each transformation to move the corresponding operator through any factor that may be present on the left of the fermion until it acts on the fermion itself. This last method, as we will see by going back to the transformation of each component of ψ , is the only correct one.

- crude : $PCT \cdot \psi_D = U_P U_C U_T \psi_D = (i\gamma^0) \gamma^2 (i\gamma^3 \gamma^1) \psi_D = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi_D$.
- “Landau” : $PCT \cdot \psi_D = P \cdot (C \cdot (T \cdot \psi_D)) = i\gamma^0 (\gamma^2 (i\gamma^3 \gamma^1 \psi_D^*)) \psi_D = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi_D$.
- cautious :

$$\begin{aligned}
\psi_D &\xrightarrow{T} T \cdot \psi_D = i\gamma^3 \gamma^1 \psi_D^* \\
&\xrightarrow{C} C \cdot (i\gamma^3 \gamma^1 \psi_D^*) \stackrel{C \text{ linear}}{=} i\gamma^3 \gamma^1 C \cdot \psi_D^* \stackrel{(16)}{=} i\gamma^3 \gamma^1 (C \cdot \psi_D)^* = i\gamma^3 \gamma^1 (\gamma^2)^* \psi_D \\
&\hspace{15em} = -i\gamma^3 \gamma^1 \gamma^2 \psi_D \\
&\xrightarrow{P} P \cdot (-i\gamma^3 \gamma^1 \gamma^2 \psi_D) \stackrel{P \text{ linear}}{=} -i\gamma^3 \gamma^1 \gamma^2 P \cdot \psi_D = -i\gamma^3 \gamma^1 \gamma^2 (i\gamma^0 \psi_D) = \gamma^3 \gamma^1 \gamma^2 \gamma^0 \psi_D \\
&\hspace{15em} = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi_D.
\end{aligned}$$

Similarly, when calculating the action of $(PCT)^2$, one gets:

- crude : $(PCT)^2 \psi_D = (-\gamma^0 \gamma^1 \gamma^2 \gamma^3) (-\gamma^0 \gamma^1 \gamma^2 \gamma^3) \psi_D = -\psi_D$.
- “Landau” : $(PCT)^2 \cdot \psi_D = PCT \cdot (PCT \cdot \psi_D) = (\gamma^0 \gamma^1 \gamma^2 \gamma^3) (\gamma^0 \gamma^1 \gamma^2 \gamma^3) \psi_D = -\psi_D$.
- cautious :

$$\begin{aligned}
(PCT)^2 \cdot \psi_D &= (PCT) \cdot ((PCT) \cdot \psi) \\
&= (PCT) \cdot (-\gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi_D) \\
&\stackrel{PCT \text{ antilinear}}{=} (-\gamma^0 \gamma^1 \gamma^2 \gamma^3)^* (PCT) \cdot \psi_D \\
&= (-\gamma^0 \gamma^1 \gamma^2 \gamma^3)^* (-\gamma^0 \gamma^1 \gamma^2 \gamma^3) \psi_D \\
&= \psi_D.
\end{aligned}$$

The “cautious” method is the only one which agrees with that directly inferred from transforming directly Weyl spinors according to the rules given in the core of the paper. One nevertheless gets the correct sign for PCT (though not for $(PCT)^2$) by the crude calculation. So, in order to discriminate without any ambiguity between the three ways of manipulating the symmetry operators when acting on a Dirac fermion, *i.e.* to avoid (or minimize) any risk of accidental agreement due to the cancellation of two mistakes, we calculated the other possible products of two operators, and compared the results with the (reliable) ones obtained when acting directly on Weyl fermions. The results are summarized below :

	TP	TC	CP
Crude (trivial product of U 's)	$\xi^\alpha \rightarrow -(\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow (\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = TC$	$\xi^\alpha \rightarrow -(\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow -(\eta^{\dot{\alpha}})^*$ $PC = CP$
“Landau” (composition)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = -TP$	$\xi^\alpha \rightarrow \eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow -\xi^\alpha$ $CT = TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$
Cautious (our way of computing)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = -TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$
Correct result (acting directly on Weyl fermions)	$\xi^\alpha \rightarrow (\eta^{\dot{\alpha}})^*$ $\eta_{\dot{\alpha}} \rightarrow -(\xi_\alpha)^*$ $PT = TP$	$\xi^\alpha \rightarrow -\eta_{\dot{\alpha}}$ $\eta_{\dot{\alpha}} \rightarrow \xi^\alpha$ $CT = -TC$	$\xi^\alpha \rightarrow (\xi_\alpha)^*$ $\eta_{\dot{\alpha}} \rightarrow (\eta^{\dot{\alpha}})^*$ $PC = CP$

Moreover, our way of computing ensures that $T^2 = 1$, in agreement with the result obtained when acting directly on Weyl spinors, while one faces problems with the Landau method which leads to $T^2 = -1$. Indeed, $T^2 \cdot \psi_D = T \cdot (i\gamma^3\gamma^1\psi_D^*) \stackrel{T \text{ antilinear}}{=} -i\gamma^3\gamma^1 T \cdot \psi_D^* \stackrel{(16)}{=} -i\gamma^3\gamma^1 (T \cdot \psi_D)^* = -i\gamma^3\gamma^1 (-i)\gamma^3\gamma^1 \psi_D = \psi_D$, while “Landau’s” prescription leads to $T^2 \cdot \psi_D = i\gamma^3\gamma^1 (i\gamma^3\gamma^1\psi_D^*)^* = i\gamma^3\gamma^1 (-i)\gamma^3\gamma^1 \psi_D = \gamma^3\gamma^1\gamma^3\gamma^1 \psi_D = -\psi_D$.

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